



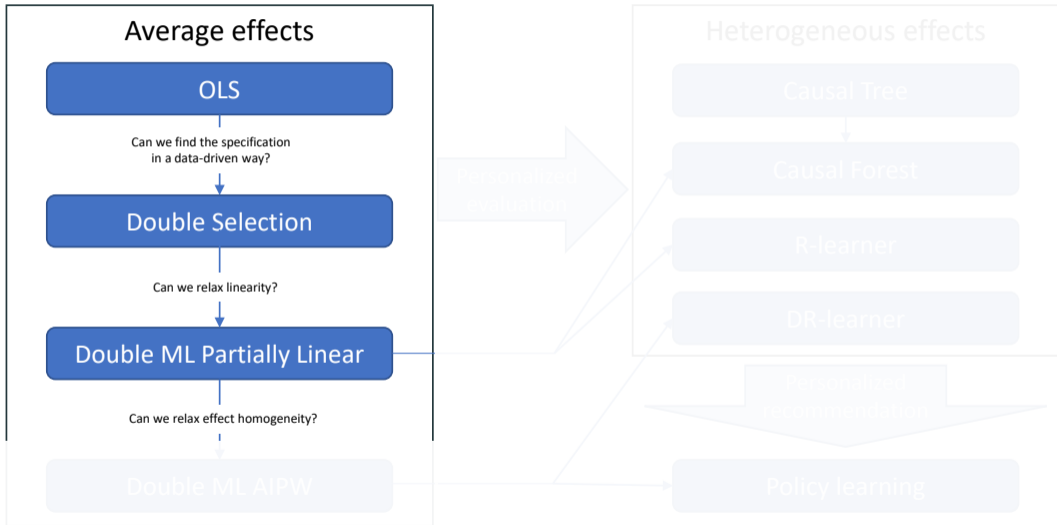
Causal Machine Learning

Average treatment effect estimation: AIPW-Double ML

Michael Knaus

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Current state of affairs



Can we allow for heterogeneous treatment effects?

1. Nonparametric identification

2. Augmented IPW

3. Consolidation

Nonparametric identification

Beyond effect homogeneity

So far, we assumed that potential outcomes are (partially) linear \Rightarrow effect heterogeneity restricted to a minimum

This resulted in relatively easy recipes for binary and continuous treatments

This is convenient

However, assuming homogeneous effects is not innocent:

- We might estimate effects for a **strange/unintuitive target population** if there is actually effect heterogeneity (e.g. [Słoczyński, 2022](#))
- We might miss that we have no comparable units in treated and control group and heavily **rely on extrapolation**

Let's see how we can relax effect homogeneity for a binary treatment

Recall target parameters

Besides SUTVA, we impose no modelling assumptions

Average target parameters:

- AVERAGE POTENTIAL OUTCOME (APO): $\gamma_w := \mathbb{E}[Y(w)]$
 - What is the expected **outcome if everybody receives treatment w** ?
- AVERAGE TREATMENT EFFECT (ATE): $\tau_{ATE} := \mathbb{E}[Y(1) - Y(0)] = \gamma_1 - \gamma_0$
 - What is the expected treatment **effect in the population**?

Definition 

Nonparametric identification (1/4)

Identifying Assumption 2 (Strong Ignorability)

(a) $Y(w) \perp\!\!\!\perp W \mid X$ for all $W \in \{0, 1\}$

(b) $0 < \mathbb{P}[W = 1 \mid X = x] := e(x) < 1$

IA2a is identical to measured confounding (IA1) of last week but here we focus on binary treatments

IA2b is called common support/overlap/positivity assumption

It is required b/c we **do not impose an outcome model** that allows to extrapolate the counterfactual into regions where everybody receives the same treatment

This set of assumptions allows for **arbitrary effect heterogeneity**

Non-parametric identification (2/4)

Note that the target parameters are just **different aggregations** of the **CONDITIONAL AVERAGE POTENTIAL OUTCOME (CAPO)** $\mathbb{E}[Y(w) | X]$:

- $\gamma_0 := \mathbb{E}[Y(0)] \stackrel{LIE}{=} \mathbb{E}[\mathbb{E}[Y(0) | X]]$
- $\gamma_1 := \mathbb{E}[Y(1)] \stackrel{LIE}{=} \mathbb{E}[\mathbb{E}[Y(1) | X]]$
- $\tau_{ATE} := \mathbb{E}[Y(1) - Y(0)] \stackrel{LIE}{=} \mathbb{E}[\mathbb{E}[Y(1) | X] - \mathbb{E}[Y(0) | X]]$

⇒ It suffices to show that the CAPO is identified

Non-parametric identification (3/4)

Three common ways to identify the CAPO under IA2:

$$\mathbb{E}[Y(w) | X = x] = \mathbb{E}[Y | W = w, X = x] =: m(w, x) \quad (1)$$

$$= \mathbb{E}\left[\frac{\mathbb{1}[W = w]Y}{e_w(x)} \middle| X = x\right] \quad (2)$$

$$= \mathbb{E}\left[m(w, x) + \frac{\mathbb{1}[W = w](Y - m(w, x))}{e_w(x)} \middle| X = x\right] \quad (3)$$

where $e_w(x) := P[W = w | X = x]$

(1) motivates estimation via regression adjustment, (2) motivates inverse probability weighting (IPW), and (3) motivates the doubly robust/augmented IPW (AIPW) estimator

Non-parametric identification (4/4)

From an identification perspective, we can plug-in any of the identified estimands

For example:

$$\begin{aligned}\gamma_w &:= \mathbb{E}[Y(w)] \stackrel{LIE}{=} \mathbb{E}[\mathbb{E}[Y(w) | X]] = \mathbb{E}[m(w, X)] \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{\mathbf{1}[W = w]Y}{e_w(X)} \middle| X\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[m(w, X) + \frac{\mathbf{1}[W = w](Y - m(w, X))}{e_w(X)} \middle| X\right]\right]\end{aligned}$$

⇒ identification 

Identification of Conditional Average Potential Outcomes - RA

Show for (1)

$$\begin{aligned}\mathbb{E}[Y(w) | X = x] &\stackrel{IA2}{=} \mathbb{E}[Y(w) | W = w, X = x] \\ &\stackrel{Consistency}{=} \mathbb{E}[Y | W = w, X = x]\end{aligned}$$

Remark: Note that the common support condition A3b is required, although it might not be obvious. It ensures that we do **not condition on an event with zero probability density**.

Identification of Conditional Average Potential Outcomes - IPW

Show for (2) by continuing and defining $D(w) := \mathbb{1}[W = w]$ for compactness

$$\begin{aligned}\mathbb{E}[Y(w) | X = x] &= \mathbb{E}[Y | W = w, X = x] \\ &= \mathbb{E}[\underbrace{D(w)}_{=1} Y | W = w, X = x] \\ &= \mathbb{E} \left[D(w) Y \frac{e_w(x)}{e_w(x)} \middle| W = w, X = x \right] + \underbrace{(1 - e_w(x)) \mathbb{E}[\underbrace{D(w)}_{=0} Y | W \neq w, X = x]}_{=0} / e_w(x) \\ &= \frac{\overbrace{e_w(x) \mathbb{E}[D(w)Y | W = w, X = x] + (1 - e_w(x)) \mathbb{E}[D(w)Y | W \neq w, X = x]}^{\stackrel{LIE}{=} \mathbb{E}[D(w)Y | X = x]}}{e_w(x)} \\ &= \frac{\mathbb{E}[D(w)Y | X = x]}{e_w(x)} = \mathbb{E} \left[\frac{D(w)Y}{e_w(x)} \middle| X = x \right]\end{aligned}$$

Identification of Conditional Average Potential Outcomes - IPW (Bonus)

A more compact derivation of IPW is to note that

$$\begin{aligned}\mathbb{E}[D(w)Y \mid X = x] &\stackrel{LIE}{=} e_w(x) \mathbb{E}[D(w)Y \mid W = w, X = x] + \overbrace{(1 - e_w(x)) \mathbb{E}[D(w)Y \mid W \neq w, X = x]}^{=0} \\ &= e_w(x) \mathbb{E}[Y \mid W = w, X = x] \\ &= e_w(x) \mathbb{E}[Y(w) \mid W = w, X = x] && \text{(SUTVA-C)} \\ &\stackrel{IA2a}{=} e_w(x) \mathbb{E}[Y(w) \mid X = x] && \mid \div e_w(x) \\ \mathbb{E}\left[\frac{D(w)Y}{e_w(x)} \mid X = x\right] &\stackrel{IA2b}{=} \mathbb{E}[Y(w) \mid X = x]\end{aligned}$$

SUTVA-C imposes SUTVA and consistency at the same step

Identification of Conditional Average Potential Outcomes - AIPW (1/2)

Define $D(w) := \mathbb{1}[W = w]$ for compactness and show for (3)

$$\begin{aligned}\mathbb{E}[Y(w) | X = x] &= \mathbb{E} \left[m(w, x) + \frac{D(w)(Y - m(w, x))}{e_w(x)} \middle| X = x \right] \\ &= \mathbb{E} \left[Y(w) - Y(w) + m(w, x) + \frac{D(w)(Y - m(w, x))}{e_w(x)} \middle| X = x \right] \\ &\stackrel{\text{SUTVA-C}}{=} \mathbb{E} \left[Y(w) - Y(w) + m(w, x) + \frac{D(w)(Y(w) - m(w, x))}{e_w(x)} \middle| X = x \right] \\ &= \mathbb{E} [Y(w) | X = x] + \underbrace{\mathbb{E} \left[(Y(w) - m(w, x)) \left(\frac{D(w) - e_w(x)}{e_w(x)} \right) \middle| X = x \right]}_{\text{needs to be 0}}\end{aligned}\quad (4)$$

Remark: Note that the common support condition IA2b is required for $m(w, x)$ to be defined and to not divide by zero as it ensures $e_w(x) > 0$

Identification of Conditional Average Potential Outcomes - AIPW (2/2)

Show that the second part of (4) is zero

$$\begin{aligned} & E \left[(Y(w) - m(w, x)) \left(\frac{D(w) - e_w(x)}{e_w(x)} \right) \middle| X = x \right] \\ & \stackrel{IA2a}{=} E [(Y(w) - m(w, x)) | X = x] E \left[\left(\frac{D(w) - e_w(x)}{e_w(x)} \right) \middle| X = x \right] \\ & = (E [Y(w) | X = x] - m(w, x)) \left(\frac{E [D(w) | X = x] - e_w(x)}{e_w(x)} \right) \\ & \stackrel{IA2, \text{SUTVA-C}}{=} (E [Y | W = w, X = x] - m(w, x)) \left(\frac{E [D(w) | X = x] - e_w(x)}{e_w(x)} \right) \\ & = \underbrace{(m(w, x) - m(w, x))}_{=0} \underbrace{\left(\frac{e_w(x) - e_w(x)}{e_w(x)} \right)}_{=0} = 0 \quad \square \end{aligned}$$

Remark: Note that only one of the nuisance parameters needs to be correct \Rightarrow doubly robust identification

Augmented IPW

The identification results suggest the following estimators for APO:

$$\text{Eq. (1): } \hat{\gamma}_w^{RA} = \frac{1}{N} \sum_i \hat{m}(w, X_i) \quad (5)$$

$$\text{Eq. (2): } \hat{\gamma}_w^{IPW} = \frac{1}{N} \sum_i \frac{\mathbb{1}[W_i = w] Y_i}{\hat{e}_w(X_i)} \quad (6)$$

$$\text{Eq. (3): } \hat{\gamma}_w^{AIPW} = \frac{1}{N} \sum_i \left(\hat{m}(w, X_i) + \frac{\mathbb{1}[W_i = w] (Y_i - \hat{m}(w, X_i))}{\hat{e}_w(X_i)} \right) \quad (7)$$

where we estimated the nuisance parameters $\hat{m}(w, x)$ and $\hat{e}_w(x)$ in a first step

Only one strategy works with ML

Estimators using **parametrically estimated nuisance parameters** are common and work for **all three strategies**

BUT model selection problem remains

⇒ Supervised ML could be helpful again

However, estimators based on only **one nuisance parameter** inherit the slow **convergence** rates of the ML method ⇒ **bias** ⇒ **no valid inference**

Chernozhukov et al. (2018) show that the estimator in (7) is **consistent**, **asymptotically normal** and **semiparametrically efficient** if nuisance parameters are **high-quality** and **cross-fitted predictions** (like last week)

AIPW Double ML: procedure

AIPW Double ML proceeds as follows:

1. Form **cross-fitted predictions** of $\hat{m}(w, X)$ and $\hat{e}_w(X)$ using ML methods
2. Create a **pseudo-outcome**

$$\tilde{Y}_{\gamma_w} = \underbrace{\hat{m}(w, X)}_{\text{outcome prediction}} + \underbrace{\frac{\mathbb{1}[W = w](Y - \hat{m}(w, X))}{\hat{e}_w(X)}}_{\text{weighted residual}} \quad (8)$$

3. Estimate APO as **mean** of the pseudo-outcome $\hat{\gamma}_w^{AIPW} = \frac{1}{N} \sum_i \tilde{Y}_{i, \gamma_w}$
4. Run **t-test** on the mean for hypothesis testing (no adjustments needed)

This is known as the **Doubly Robust or AUGMENTED INVERSE PROBABILITY WEIGHTING** (AIPW) estimator (I prefer AIPW b/c there are more doubly robust estimators than there are more AIPW estimators, but never mind...)

Why does it work?

The "magic" feature is again that the underlying score

$$\frac{1}{N} \sum_{i=1}^N \underbrace{\left(\hat{m}(w, X_i) + \frac{\mathbb{1}[W_i = w](Y_i - \hat{m}(w, X_i))}{\hat{e}_w(X_i)} - \hat{\gamma}_w^{AIPW} \right)}_{\psi(Y_i, W_i, \hat{m}(w, X_i), \hat{e}_w(X_i))} = 0 \quad (9)$$

$$\Rightarrow \hat{\gamma}_w^{AIPW} = \frac{1}{N} \sum_{i=1}^N \left(\hat{m}(w, X_i) + \frac{\mathbb{1}[W_i = w](Y_i - \hat{m}(w, X_i))}{\hat{e}_w(X_i)} \right) \quad (10)$$

is **Neyman-orthogonal**, i.e. small errors in the estimation of nuisance parameters do not distort estimation of target parameter

Estimators based on regression adjustment or IPW only are not Neyman-orthogonal

Neyman-orthogonality of AIPW (1/5)

Neyman-orthogonality means that the Gateaux derivative with respect to the nuisance parameters is zero in expectation at the true nuisance parameters (NP):

$$\partial_r \mathbb{E}[\psi(\dots, m + r(\tilde{m} - m), e + r(\tilde{e} - e))] |_{r=0} = 0 \quad (11)$$

where we suppress the dependencies of NPs and denote by, e.g., \tilde{m} any other value of the outcome nuisance than the true value m

This looks scary, but we just need to know how to setup the problem and take standard derivatives (quotient rule)

For simplicity, we get rid of the brackets and the underscore to write the score with true target and nuisance parameters as

$$\psi(Y, W, \tau, m(w, X), e(X)) = m + \frac{\mathbb{1}[W = w]Y}{e} - \frac{\mathbb{1}[W_i = w]m}{e} - \gamma_w$$

Neyman-orthogonality of AIPW (2/5)

Again we use $D(w) := \mathbb{1}[W = w]$ for brevity

First, add perturbations to the true nuisance parameters in the score

$$\begin{aligned} & \psi(\dots, m + r(\tilde{m} - m), e + r(\tilde{e} - e)) \\ &= (m + r(\tilde{m} - m)) + \frac{D(w)Y}{e + r(\tilde{e} - e)} - \frac{D(w)(m + r(\tilde{m} - m))}{e + r(\tilde{e} - e)} - \gamma_w \\ &= \underbrace{(m + r(\tilde{m} - m))}_{(i)} + \underbrace{\frac{D(w)Y}{e + r(\tilde{e} - e)}}_{(ii)} - \underbrace{\frac{D(w)m}{e + r(\tilde{e} - e)}}_{(iii)} - \underbrace{\frac{D(w)r(\tilde{m} - m)}{e + r(\tilde{e} - e)}}_{(iv)} - \gamma_w \end{aligned}$$

Note that with $r = 0$, we are back to the original score

With $r \neq 0$ the nuisance parameters are distorted

Next, take the derivative wrt r

Neyman-orthogonality of AIPW (3/5)

Second, take the derivative wrt r

$$\begin{aligned} & \partial_r \psi(\dots, m + r(\tilde{m} - m), e + r(\tilde{e} - e)) \\ &= \underbrace{(\tilde{m} - m)}_{\partial_r(i)} - \underbrace{\frac{D(w)Y(\tilde{e} - e)}{(e + r(\tilde{e} - e))^2}}_{\partial_r(ii)} + \underbrace{\frac{D(w)m(\tilde{e} - e)}{(e + r(\tilde{e} - e))^2}}_{\partial_r(iii)} \\ & \quad - \underbrace{\frac{D(w)(\tilde{m} - m)(e + r(\tilde{e} - e)) - D(w)r(\tilde{m} - m)(\tilde{e} - e)}{(e + r(\tilde{e} - e))^2}}_{\partial_r(iv)} \end{aligned}$$

Neyman-orthogonality of AIPW (4/5)

Third, evaluate at $r = 0$

$$\begin{aligned} & \partial_r \psi(\dots, m + r(\tilde{m} - m), e + r(\tilde{e} - e))|_{r=0} \\ &= (\tilde{m} - m) - \frac{D(w)Y(\tilde{e} - e)}{(e + \mathbf{0}(\tilde{e} - e))^2} + \frac{D(w)m(\tilde{e} - e)}{(e + \mathbf{0}(\tilde{e} - e))^2} \\ & \quad - \frac{D(w)(\tilde{m} - m)(e + \mathbf{0}(\tilde{e} - e)) - D(w)\mathbf{0}(\tilde{m} - m)(\tilde{e} - e)}{(e + \mathbf{0}(\tilde{e} - e))^2} \\ &= (\tilde{m} - m) - \frac{D(w)Y(\tilde{e} - e)}{e^2} + \frac{D(w)m(\tilde{e} - e)}{e^2} - \frac{D(w)(\tilde{m} - m)e}{e^2} \end{aligned}$$

Neyman-orthogonality of AIPW (5/5)

Fourth, take expectation

$$\begin{aligned} & \partial_r \mathbb{E}[\psi(\dots, m + r(\tilde{m} - m), e + r(\tilde{e} - e))] |_{r=0} \\ &= \mathbb{E} \left[(\tilde{m} - m) - \frac{D(w)Y(\tilde{e} - e)}{e^2} + \frac{D(w)m(\tilde{e} - e)}{e^2} - \frac{D(w)(\tilde{m} - m)e}{e^2} \right] \\ &\stackrel{LIE}{=} \mathbb{E} \left[\mathbb{E} \left[(\tilde{m} - m) - \frac{D(w)Y(\tilde{e} - e)}{e^2} + \frac{D(w)m(\tilde{e} - e)}{e^2} - \frac{D(w)(\tilde{m} - m)e}{e^2} \middle| X \right] \right] \\ &= \mathbb{E} \left[(\tilde{m} - m) - \frac{em(\tilde{e} - e)}{e^2} + \frac{em(\tilde{e} - e)}{e^2} - \frac{e(\tilde{m} - m)e}{e^2} \right] = 0 \end{aligned}$$

because

$$\begin{aligned} \mathbb{E}[D(w)Y | X] &\stackrel{LIE}{=} \mathbb{P}[D(w) = 0 | X] \mathbb{E}[0Y | W \neq w, X] + \mathbb{P}[D(w) = 1 | X] \mathbb{E}[1Y | W = w, X] \\ &= \mathbb{P}[D(w) = 1 | X] \mathbb{E}[Y | W = w, X] = em \end{aligned}$$

⇒ The Gateaux derivative wrt NP is zero ⇒ Neyman-orthogonal score

Average treatment effect (1/2)

Following the same logic we can estimate the ATE as follows:

1. Form **cross-fitted predictions** of $\hat{m}(1, X)$, $\hat{m}(0, X)$ and $\hat{e}(X)$ using ML methods
2. Create a **pseudo-outcome**

$$\begin{aligned}\tilde{Y}_{ATE} &= \tilde{Y}_{\gamma_1} - \tilde{Y}_{\gamma_0} && (12) \\ &= \underbrace{\hat{m}(1, X) - \hat{m}(0, X)}_{\text{outcome predictions}} + \underbrace{\frac{W(Y - \hat{m}(1, X))}{\hat{e}(X)} - \frac{(1 - W)(Y - \hat{m}(0, X))}{1 - \hat{e}(X)}}_{\text{weighted residuals}}\end{aligned}$$

where we use that $\mathbb{1}[W = 1] = W$, $\mathbb{1}[W = 0] = 1 - W$, $e_1(X) = e(X)$,
 $e_0(X) = 1 - e(X)$

3. Estimate ATE as **mean** of the pseudo-outcome $\hat{\tau}_{ATE}^{AIPW} = \frac{1}{N} \sum_i \tilde{Y}_{i,ATE}$
4. Run **t-test** on the mean for hypothesis testing (no adjustments needed)

Average treatment effect (2/2)

As pseudo-outcome (12) is just the difference of the APO pseudo outcomes it inherits the Neyman-orthogonality

$\Rightarrow \hat{\tau}_{ATE}^{AIPW}$ is consistent, asymptotically normal and semiparametrically efficient when nuisance parameters are high-quality and cross-fitted predictions

Estimation 

Advantages:

- No need to impose effect homogeneity \Rightarrow differentiate between ATE and ATT (next lecture)
- Extends naturally to multiple treatments
- Basis for other estimators (see soon)

Disadvantages:

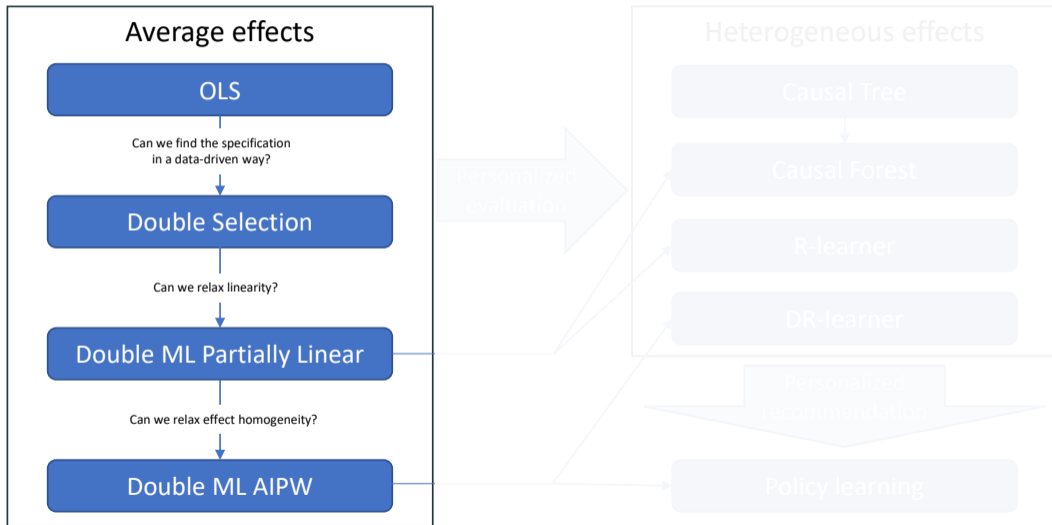
- Looks scary and complicated
- Sensitive to small propensity scores (no/weak overlap)
- Extensions for continuous treatments not trivial (Colangelo & Lee, 2019; Semenova & Chernozhukov, 2021)

Simulation notebook: AIPW Double ML (ATE)

Application notebook: Double ML for average treatment effects

Consolidation

Average effects unlocked



Main take-away

Estimation of average treatment effects can be split into multiple prediction problems

Combining them in the right way allows to use familiar inference procedures

⇒ We can leverage the powerful supervised ML toolbox

⇒ Moves the academic task from hand crafted regression models to the specification of suitable prediction methods

⇒ In the best case this increases

- quality and statistical validity of the estimates
- transparency
- time for researchers to do something more interesting

The recipe of Double ML can be generalized ⇒ next week

Ceterum censeo a fancy method alone is not a credible
identification strategy
⇒ separate identification and estimation