Supplemental Material: Probabilistic Gas Leak Rate Estimation using Submodular Function Maximization with Routing Constraints

Kalvik Jakkala and Srinivas Akella

I. Derivation of EER and Posterior for Gaussian prior

We use the second order Rényi entropy function in our Bayesian model.

$$H(R) = -\log_2 \int_{r \epsilon R} p^2(r) dr \tag{1}$$

The conditional probability of a gas concentration given a leak rate is defined as follows. We marginalize this term to get the probability distribution of gas concentration m.

$$p(s|m) = \frac{p(m|s)p(s)}{\int_{s_1 \epsilon S} p(m|s_1)p(s_1)ds_1} \tag{2}$$

$$p(m) = \int_{s \epsilon S} p(m,s)ds = \int_{s \epsilon S} p(m|s)p(s)ds \tag{3}$$

Albertson et al. [1] modeled the distribution of a leak rate across data collection tours with the following wherein j is the tour number. Furthermore, we model the initial prior as a Gaussian as shown below.

$$\begin{aligned} p(s) &= \begin{cases} p(s), & \text{for } j = 1 \\ p(s|m)_{j-1}, & \text{for } j > 1 \end{cases} \\ p(s) &\sim \mathcal{N}(\mu_s, \sigma_s^2) \end{aligned} \tag{4}$$

We, consider the gas dispersion model of Foster et al. [2] in our derivation. This allows us to model the conditional of a gas concentration rate given a leak rate in terms of the dispersion model. Moreover, we assume the dispersion model to be linear in the leak rate.

$$C(s,x,y,z) = s \underbrace{\frac{1}{U}\left(\frac{\bar{A}}{\bar{z}(x)} \exp\left[-\left(\frac{Bz}{\bar{z}(x)}\right)^2\right]\right)\left(\frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2\right]\right)}_{constant,\ independent\ of\ function\ input\ 's'} = s\mathcal{A}(x,y,z) \tag{5}$$

$$\begin{aligned} p(m|s) &= \frac{1}{\sigma_e\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{m - C(s,x,y,z)}{\sigma_e}\right)^2\right] \\ &= \frac{1}{\sigma_e\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{m - s\mathcal{A}(x,y,z)}{\sigma_e}\right)^2\right] \\ p(m|s) &\sim \mathcal{N}(s\mathcal{A}, \sigma_e^2) \end{aligned} \tag{6}$$

The authors are with the Computer Science Department, University of North Carolina at Charlotte, Charlotte, NC, USA. Email:{kjakkala, sakella}@uncc.edu

This work was funded in part by the NSF Award IIP-1919233.

$$
\begin{aligned}
p(m|s)p(s) &= \mathcal{N}(s\mathcal{A},\sigma_e^2)\mathcal{N}(\mu_s,\sigma_s^2) \\
&= \frac{1}{\sigma_e\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{m-s\mathcal{A}}{\sigma_e}\right)^2\right]\frac{1}{\sigma_s\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{1}{2}\left(\left(\frac{m-s\mathcal{A}}{\sigma_e}\right)^2+\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right)\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{1}{2}\left(\frac{m^2-2ms\mathcal{A}+s^2\mathcal{A}^2}{\sigma_e^2}+\frac{s^2-2s\mu_s+\mu_s^2}{\sigma_s^2}\right)\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\left(\frac{m^2\sigma_s^2-2ms\mathcal{A}\sigma_s^2+s^2\mathcal{A}^2\sigma_s^2+s^2\sigma_e^2-2s\mu_s\sigma_e^2+\mu_s^2\sigma_e^2}{2\sigma_e^2\sigma_s^2}\right)\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\left(\frac{s^2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)-2s(m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2)+(\mu_s^2\sigma_e^2+m^2\sigma_s^2)}{2\sigma_e^2\sigma_s^2}\right)\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{s^2-2s\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}+\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{s^2-2s\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}+\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}+\frac{\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2-\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{s^2-2s\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}+\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}+\frac{\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}-\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}+\frac{\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}-\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]\exp\left[\underbrace{\frac{\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}-\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}}_{\Lambda}\right] \\
&= \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]\exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right] \qquad \text{From (12)}
\end{aligned} \tag{7}
$$

The definite integral of an arbitrary Gaussian function

$$
\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}} \tag{8}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} p^2(s)dx &= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma_s^2}}e^{\frac{-(x-\mu_s)^2}{2\sigma_s^2}}\right)^2dx \\
&= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_s^2}e^{\frac{-\cancel{2}(x-\mu_s)^2}{\cancel{2}\sigma_s^2}}dx \\
&= \frac{1}{2\pi\sigma_s^2}\int_{-\infty}^{\infty}e^{\frac{-(x-\mu_s)^2}{\sigma_s^2}}dx \\
&= \frac{\sigma_s\sqrt{\pi}}{2\pi\sigma_s^2} \qquad \text{From (8)} \\
&= \frac{1}{2\sigma_s\sqrt{\pi}}
\end{aligned} \tag{9}
$$

$$\begin{aligned}
\int_{s\epsilon S} p(m|s)p(s)ds &= \int_{s\epsilon S} \frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]\exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]ds \\
&= \exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{2\pi\sigma_e\sigma_s}\int_{s\epsilon S}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]ds \\
&= \exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{2\pi\sigma_e\sigma_s}\sqrt{\pi\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}} \qquad \text{From (8)}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\int_{s\epsilon S} (p(m|s)p(s))^2 ds &= \int_{s\epsilon S}\left(\frac{1}{2\pi\sigma_e\sigma_s}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]\exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\right)^2 ds \\
&= \left(\exp\left[\frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{2\pi\sigma_e\sigma_s}\right)^2\int_{s\epsilon S}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]^2 ds \\
&= \exp\left[\frac{\cancel{2}(m-\mu_s\mathcal{A})^2}{\cancel{2}(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{4\pi^2\sigma_e^2\sigma_s^2}\int_{s\epsilon S}\exp\left[-\cancel{2}\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\cancel{2}\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]ds \\
&= \exp\left[\frac{(m-\mu_s\mathcal{A})^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{4\pi^2\sigma_e^2\sigma_s^2}\int_{s\epsilon S}\exp\left[-\frac{\left(s-\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}\right]ds \\
&= \exp\left[\frac{(m-\mu_s\mathcal{A})^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}\right]\frac{1}{4\pi^2\sigma_e^2\sigma_s^2}\sqrt{\pi\frac{\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}} \qquad \text{From (8)}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\Lambda &= \frac{\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}-\left(\frac{m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}\right)^2}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}} \\
&= \frac{\frac{\mu_s^2\sigma_e^2+m^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}-\frac{(m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2)^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)^2}}{\frac{2\sigma_e^2\sigma_s^2}{\mathcal{A}^2\sigma_s^2+\sigma_e^2}} \\
&= \frac{\frac{(\mu_s^2\sigma_e^2+m^2\sigma_s^2)(\mathcal{A}^2\sigma_s^2+\sigma_e^2)-(m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2)^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)^{\cancel{2}}}}{\frac{2\sigma_e^2\sigma_s^2}{\cancel{\mathcal{A}^2\sigma_s^2+\sigma_e^2}}} \\
&= \frac{(\mu_s^2\sigma_e^2+m^2\sigma_s^2)(\mathcal{A}^2\sigma_s^2+\sigma_e^2)-(m\mathcal{A}\sigma_s^2+\mu_s\sigma_e^2)^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)(2\sigma_e^2\sigma_s^2)} \\
&= \frac{\mu_s^2\sigma_e^2\mathcal{A}^2\sigma_s^2+\cancel{\mu_s^2\sigma_e^4}+\cancel{m^2\sigma_s^4\mathcal{A}^2}+m^2\sigma_s^2\sigma_e^2-\cancel{m^2\mathcal{A}^2\sigma_s^4}-\cancel{\mu_s^2\sigma_e^4}-2m\mathcal{A}\sigma_s^2\mu_s\sigma_e^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)(2\sigma_e^2\sigma_s^2)} \\
&= \frac{\mu_s^2\sigma_e^2\mathcal{A}^2\sigma_s^2+m^2\sigma_s^2\sigma_e^2-2m\mathcal{A}\sigma_s^2\mu_s\sigma_e^2}{(\mathcal{A}^2\sigma_s^2+\sigma_e^2)(2\sigma_e^2\sigma_s^2)} \\
&= \frac{\cancel{(\sigma_e^2\sigma_s^2)}(\mu_s^2\mathcal{A}^2+m^2-2m\mathcal{A}\mu_s)}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)(\cancel{\sigma_e^2\sigma_s^2})} \\
&= \frac{(\mu_s^2\mathcal{A}^2+m^2-2m\mathcal{A}\mu_s)}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)} \\
&= \frac{(m-\mu_s\mathcal{A})^2}{2(\mathcal{A}^2\sigma_s^2+\sigma_e^2)}
\end{aligned} \tag{12}$$

The EER was derived as follows:

$$
\begin{aligned}
\varphi[S;M] &= H(S) - \int\limits_{m\epsilon M} H(S|m)p(m)dm \\
&= -\log_2 \int\limits_{s\epsilon S} p^2(s)ds + \int\limits_{m\epsilon M} \log_2 \int\limits_{s\epsilon S} p^2(s|m)ds \; p(m)dm && \text{From (1)} \\
&= -\log_2 \int\limits_{s\epsilon S} p^2(s)ds + \int\limits_{m\epsilon M} \log_2 \int\limits_{s\epsilon S} \left(\frac{p(m|s)p(s)}{\int_{s_1\epsilon S} p(m|s_1)p(s_1)ds_1} \right)^2 ds \; p(m)dm && \text{From (2)} \\
&= -\log_2 \int\limits_{s\epsilon S} p^2(s)ds + \int\limits_{m\epsilon M} \log_2 \frac{\int_{s\epsilon S}(p(m|s)p(s))^2 ds}{(\int_{s_1\epsilon S} p(m|s_1)p(s_1)ds_1)^2} \; p(m)dm \\
&= -\log_2 \int\limits_{s\epsilon S} p^2(s)ds + \int\limits_{m\epsilon M} \log_2 \frac{\int_{s\epsilon S}(p(m|s)p(s))^2 ds}{(\int_{s_1\epsilon S} p(m|s_1)p(s_1)ds_1)^{\cancel{2}}} \; \cancel{\int\limits_{s_2\epsilon S} p(m|s_2)p(s_2)ds_2} \; dm && \text{From (3)} \\
&= -\log_2 \int\limits_{s\epsilon S} p^2(s)ds + \int\limits_{m\epsilon M} \log_2 \frac{\int_{s\epsilon S}(p(m|s)p(s))^2 ds}{(\int_{s_1\epsilon S} p(m|s_1)p(s_1)ds_1)} \; dm \\
&= -\log_2 \left(\frac{1}{2\sigma_s\sqrt{\pi}} \right) + \int\limits_{m\epsilon M} \log_2 \left(\frac{\exp\left[\frac{(m-\mu_s A_{xyz})^2}{(A_{xyz}^2\sigma_s^2+\sigma_e^2)}\right] \frac{1}{4\pi^2\sigma_e^2\sigma_s^2} \sqrt{\pi \frac{\sigma_e^2\sigma_s^2}{A_{xyz}^2\sigma_s^2+\sigma_e^2}}}{\exp\left[\frac{(m-\mu_s A_{xyz})^2}{2(A_{xyz}^2\sigma_s^2+\sigma_e^2)}\right] \frac{1}{2\pi\sigma_e\sigma_s} \sqrt{\pi \frac{2\sigma_e^2\sigma_s^2}{A_{xyz}^2\sigma_s^2+\sigma_e^2}}} \right) dm && \text{From (10, 11)} \qquad (13) \\
&= -\log_2 \left(\frac{1}{2\sigma_s\sqrt{\pi}} \right) + \int\limits_{m\epsilon M} \log_2 \left(\exp\left[\frac{(m-\mu_s A_{xyz})^2}{2(A_{xyz}^2\sigma_s^2+\sigma_e^2)}\right] \frac{2\pi\sigma_e\sigma_s}{4\pi^2\sigma_e^2\sigma_s^2} \frac{\sqrt{\pi \frac{\sigma_e^2\sigma_s^2}{A_{xyz}^2\sigma_s^2+\sigma_e^2}}}{\sqrt{\pi \frac{2\sigma_e^2\sigma_s^2}{A_{xyz}^2\sigma_s^2+\sigma_e^2}}} \right) dm \\
&= -\log_2 \left(\frac{1}{2\sigma_s\sqrt{\pi}} \right) + \int\limits_{m\epsilon M} \log_2 \left(\frac{1}{2\sqrt{2}\pi} \exp\left[\frac{(m-\mu_s A_{xyz})^2}{2(A_{xyz}^2\sigma_s^2+\sigma_e^2)}\right] \right) dm \\
&= -\log_2 \left(\frac{1}{2\sigma_s\sqrt{\pi}} \right) + \log_2 \left(\frac{1}{2\sqrt{2}\pi} \right) + \int\limits_{m\epsilon M} \log_2 \left(\exp\left[\frac{(m-\mu_s A_{xyz})^2}{2(A_{xyz}^2\sigma_s^2+\sigma_e^2)}\right] \right) dm \\
&= -\log_2 \left(\frac{1}{2\sigma_s\sqrt{\pi}} \right) + \int\limits_{m\epsilon M} \frac{(m-\mu_s A_{xyz})^2}{2(A_{xyz}^2\sigma_s^2+\sigma_e^2)} \; dm + c
\end{aligned}
$$

Finally, we derived the posterior of the leak rate as follows.

$$
\begin{aligned}
\psi &= \left(\frac{s-\mu_s}{\sigma_s}\right)^2 + \left(\frac{M-s\mathbf{A}}{\sigma_e}\right)^2 \\
&= \frac{M^\top M - 2M^\top \mathbf{A}s + \mathbf{A}^\top \mathbf{A}s^2}{\sigma_e^2} + \frac{s^2 - 2s\mu_s + \mu_s^2}{\sigma_s^2} \\
&= \frac{M^\top M - 2M^\top \mathbf{A}s + \mathbf{A}^\top \mathbf{A}s^2 + s^2 - 2s\mu_s + \mu_s^2}{\sigma_e^2\sigma_s^2} \\
&= \frac{s^2(\mathbf{A}^\top\mathbf{A}\sigma_s^2 + \sigma_e^2) - 2s(M^\top\mathbf{A}\sigma_s^2 + \mu_s\sigma_e^2) + (\mu_s^2\sigma_e^2 + \sigma_s^2 M^\top M)}{\sigma_e^2\sigma_s^2} \\
&= \frac{s^2 - 2s\frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2} + \frac{\mu_s^2\sigma_e^2+\sigma_s^2M^\top M}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}} \\
&= \frac{\left(s - \frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}} + \frac{\frac{\mu_s^2\sigma_e^2+\sigma_s^2M^\top M}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2} - \left(\frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}} \\
p(s) &\sim \mathcal{N}(\mu_s, \sigma_s^2) \\
p(M|s) &\sim \mathcal{N}(\mathbf{A}s, \sigma_e^2 I) \\
p(s|M) &= \frac{p(s)p(M|s)}{p(M)} \\
p(s)p(M|s) &= \frac{1}{\sigma_s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{s-\mu_s}{\sigma_s}\right)^2\right] \frac{1}{\sigma_e\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{M-s\mathbf{A}}{\sigma_e}\right)^2\right] \\
&= \frac{1}{2\pi\sigma_s\sigma_e} \exp\left[-\frac{1}{2}\left(\underbrace{\left(\frac{s-\mu_s}{\sigma_s}\right)^2 + \left(\frac{M-s\mathbf{A}}{\sigma_e}\right)^2}_{\psi}\right)\right] \\
&= \frac{1}{2\pi\sigma_s\sigma_e} \exp\left[-\frac{1}{2}\left(\frac{\left(s - \frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}} + \frac{\frac{\mu_s^2\sigma_e^2+\sigma_s^2M^\top M}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2} - \left(\frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}}\right)\right] \\
&= \frac{1}{2\pi\sigma_s\sigma_e} \exp\left[-\frac{1}{2}\frac{\left(s - \frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}}\right] \exp\left[-\frac{1}{2}\frac{\frac{\mu_s^2\sigma_e^2+\sigma_s^2M^\top M}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2} - \left(\frac{M^\top\mathbf{A}\sigma_s^2+\mu_s\sigma_e^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}\right)^2}{\frac{\sigma_e^2\sigma_s^2}{\mathbf{A}^\top\mathbf{A}\sigma_s^2+\sigma_e^2}}\right]
\end{aligned}
\tag{14}
$$

REFERENCES

[1] John D Albertson, Tierney Harvey, Greg Foderaro, Pingping Zhu, Xiaochi Zhou, Silvia Ferrari, M Shahrooz Amin, Mark Modrak, Halley Brantley, and Eben D Thoma. A mobile sensing approach for regional surveillance of fugitive methane emissions in oil and gas production. *Environmental Science & Technology*, 50(5):2487–2497, 2016.

[2] Tierney A Foster-Wittig, Eben D Thoma, and John D Albertson. Estimation of point source fugitive emission rates from a single sensor time series: A conditionally-sampled gaussian plume reconstruction. *Atmospheric Environment*, 115:101–109, 2015.