Review of the Basics of Conformal Field Theory, the Conformal Bootstrap, and Chern-Simons

Karpur Shukla[1]

Department of Fluids and Thermal Sciences, School of Engineering, Brown University
182 Hope Street, Providence, Rhode Island, United States, 02912

E-mail: karpur_shukla@brown.edu

Abstract: Here, we briefly review the conformal bootstrap, a numerical technique to determine the conformal data of a conformal field theory. We discuss conformal transformations and conformal invariance; basic properties of operators; radial quantisation; bounds from unitarity; the operator product expansion and consequences of locality and crossing symmetry; and the general principle of the bootstrap itself.

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[1]ORCID: 0000-0002-7775-6979.

1 Conformal Invariance and the Conformal Group

Conformal field theories (CFTs) are, unsurprisingly, quantum field theories (QFTs) that satisfy conformal invariance. As a result, to understand the properties of CFTs, we first examine conformal invariance and its consequences. Here, we primarily follow [1–5], with some additional insights from [6–11].

1.1 Conformal Transformations and Identities

We start with the d-dimensional manifolds M and N, with metric tensors g and g respectively. If we examine differentiable maps $\varphi : U \to V$ for the open subsets $U \subset M$ and $V \subset N$, φ is a conformal transformation if it satisfies $\varphi^* g = \Lambda g$ for some positive scale factor $\Lambda \in \mathbb{R}^+$. Considering flat space with the signature $(p, q, r = 0)$, imposing $N = M$ (so that we have $g = g$), and labelling everything by their coordinates (so that $\varphi(x) \in V$ is $\varphi(x) = x'$)), the conformality requirement $\varphi^* g = \Lambda g$ is given by:

$$g_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, g_{\mu\nu} \tag{1.1}$$

(Here, the line element is $\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}$.) These correspond to *angle-preserving* maps on M; an example of this is depicted in Figure 1. (Using that example, we can see that conformal transformations preserve the angles between vectors, but not necessarily the lengths of the vectors themselves.) The Jacobian of conformal transformations is given from (1.1) by:

$$J = \left| \frac{\partial x'^{\mu}}{\partial x^{\nu}} \right| = \sqrt{\det\{\Lambda(x)\, g_{\mu\nu}\}} = \frac{1}{\Lambda^{d/2}} \tag{1.2}$$

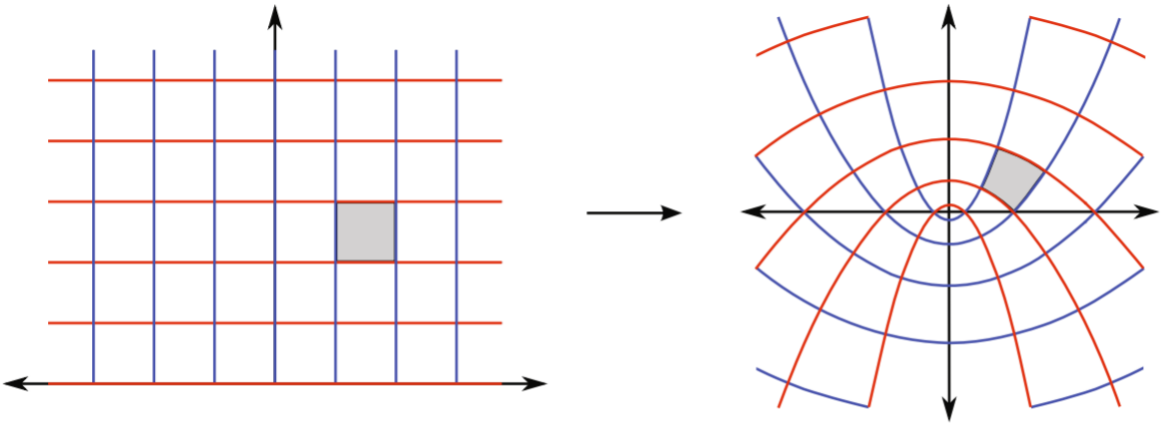

Figure 1: An example of a conformal transformation, which preserves the angles between vectors (but not necessarily the lengths of the vectors themselves). Taken from [1].

Unsurprisingly, if we impose (1.1) onto $g_{\mu\nu} \mapsto g_{\rho\sigma}$ when examining infinitesimal coordinate transformations $x^{\mu} \mapsto x^{\mu} + \epsilon^{\mu}(x) + \mathcal{O}(\epsilon^2)$ up to first order in $\epsilon(x) \ll 1$, we directly get the Cauchy-Riemann equations:

$$\partial_{\nu} \epsilon_{\mu} + \partial_{\mu} \epsilon_{\nu} = \frac{2\, \partial^{\mu} \epsilon_{\mu}}{d} = \frac{2(\partial \cdot \epsilon)}{d} g_{\mu\nu} \tag{1.3}$$

This then gives the scale factor to order $\mathcal{O}(\epsilon)$ as:

$$\Lambda(x) = 1 + \frac{2\,\partial^\mu \epsilon_\mu}{d} \tag{1.4}$$

(These expressions are derives in §A.1.) Algebraically manipulating the Cauchy-Riemann equations (1.3) give us two important identities mentioned in [1, 2], derived in §A.1[1]:

$$\big[(d-2)\,\partial_\mu \partial_\nu + g_{\mu\nu}\,\Box\big](\partial\cdot\epsilon) = 0 \quad \Longrightarrow \quad (d-1)\,\Box(\partial\cdot\epsilon) = 0 \tag{1.5a}$$

$$\partial_\mu \partial_\nu \epsilon_\rho = \frac{1}{d}\big(g_{\rho\mu}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu - g_{\mu\nu}\,\partial_\rho\big)[\partial\cdot\epsilon] \tag{1.5b}$$

We note something quite important about (1.5a), following [2]. The $d-2$ term highlights that the identity we get from further manipulations will hold specifically for $d \geq 3$. Conversely, this expression *doesn't* apply (or, more precisely, vanishes identically) for 2D CFTs; this is just one of the many ways that 2D CFTs are particularly special. ([2] also discusses the $d = 1$ case; i.e., conformal quantum mechanics (CQM).)

Finally, we also note that the Poincaré group is the semidirect product of translations and the Lorentz group: $\mathsf{ISO}(d,1) = \mathsf{T}(d) \rtimes \mathsf{O}(d,1)$. Thus, the Poincaré group is a subgroup of the conformal group; namely, Poincaré transformations have $\Lambda = 1$. As a result, conformal invariance is more restrictive than Poincaré invariance: only some QFTs satisfy conformal invariance.

1.2 The Conformal Group

The identity (1.5b) restricts $(\partial\cdot\epsilon)$ to be at most linear in x^μ, since its second derivative $\Box(\partial\cdot\epsilon)$ is zero. Thus, $(\partial\cdot\epsilon)$ (a scalar) must at *most* look like $A + B_\mu x^\mu$ (a scalar plus a covariant vector contracted with x^μ). From this restriction on $(\partial\cdot\epsilon)$, the infinitesimal transformation ϵ_μ must be at most quadratic in x^μ; i.e., at most, we have:

$$\epsilon_\mu \approx a_\mu + b_{\mu\nu}\,x^\nu + c_{\mu\nu\rho}\,x^\nu\,x^\rho \tag{1.6a}$$

(We note that ϵ_μ is a vector, so a_μ is a vector, $b_{\mu\nu}$ is a rank-2 tensor to contract with x^ν, and $c_{\mu\nu\rho}$ is rank-3 to contract with $x^\nu\,x^\rho$.) Since $c_{\mu\nu\rho}$ is contracting with $x^\nu\,x^\rho$, it's symmetric in the last two indices: $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Finally, we note that ϵ_μ is an infinitesimal transformation, so a_μ, $b_{\mu\nu}$, and $c_{\mu\nu\rho} \ll 1$.

Using (1.3) and (1.5), we can determine the overall shape of a_μ, $b_{\mu\nu}$, and $c_{\mu\nu\rho}$, and most importantly associate them with specific types of transformations. This is done in the unabridged version; here, we simply report the results.

- For a_μ, the derivation is trivial: we have $\epsilon_\mu \approx a_\mu$. Since a_μ is a constant, the derivatives all go to zero. a_μ gives the translations $x^\mu \mapsto x^\mu + a^\mu$, generated by the translation operator $\hat{P}_\mu = -\mathrm{i}\hat{\partial}_\mu$.

[1] Since we're in Minkowski space, the Riemann tensor $R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma}$ vanishes everywhere, and thus the covariant derivatives ∂_μ equal partial derivatives $\vec{\nabla}$. Thus, $[\partial_\mu, \partial_\nu] = 0$. These properties were used in the derivations in the appendix.

- For $b_{\mu\nu}$, we insert $\epsilon_\mu \approx a_\mu + b_{\mu\nu}\, x^\nu$ into (1.3), which ends up giving:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\left(g^{\rho\sigma}\, b_{\sigma\rho}\right) g_{\mu\nu} \tag{1.6b}$$

  Thus, the symmetric part of $b_{\mu\nu}$ is proportional to the metric $g_{\mu\nu}$. We can then split $b_{\mu\nu}$ into a symmetric part and an antisymmetric part, where the symmetric part is proportional to the metric:

$$b_{\mu\nu} = \alpha\, g_{\mu\nu} + m_{\mu\nu} \tag{1.6c}$$

  – The symmetric term $\alpha g_{\mu\nu}$ gives dilations $x^\mu \mapsto (1+\alpha)\, x^\mu$. These are generated by the dilation operator $\hat{D} = -\mathrm{i} x^\mu \hat{\partial}_\mu$. (Dilations are also sometimes referred to as scale transformations and dilatations.)

  – The antisymmetric term $m_{\mu\nu}$ gives rotations $x^\mu = \left(\delta^\nu_\mu + m^\mu_\nu\right) x^\mu$. These are generated by the angular momentum operator $\hat{L}_{\mu\nu} = \mathrm{i}\big(x_\mu\, \hat{\partial}_\nu - x_\nu\, \hat{\partial}_\mu\big)$.

- For $c_{\mu\nu\rho}$, we insert $\epsilon_\mu \approx a_\mu + b_{\mu\alpha}\, x^\alpha + c_{\mu\alpha\beta}\, x^\alpha\, x^\beta$ into (1.5b), which ends up giving:

$$c_{\rho\nu\mu} + c_{\rho\mu\nu} = c_{\rho\mu\nu} = \frac{g_{\rho\nu}\, g^{\alpha\beta}\, c_{\alpha\beta\mu} + g_{\nu\rho}\, g^{\alpha\beta}\, c_{\alpha\beta\mu} - g_{\mu\nu}\, g^{\alpha\beta}\, c_{\alpha\beta\rho}}{d} \tag{1.6d}$$

  Defining $b_\nu := g^{\alpha\beta}\, c_{\alpha\beta\nu}/d$, this gives:

$$c_{\rho\mu\nu} = g_{\rho\mu}\, b_\nu + g_{\nu\rho}\, b_\mu + g_{\mu\nu}\, b_\rho \tag{1.6e}$$

  $c_{\rho\mu\nu}$ gives special conformal transformations (SCTs), which we can write in terms of b_ν as:

$$x^\mu \mapsto x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)} \tag{1.6f}$$

  These are generated by the operator $\hat{K}_{\mu\nu} = -\mathrm{i}\big(2x_\mu\, x^\nu\, \hat{\partial}_\nu - (x\cdot x)\hat{\partial}_\mu\big)$.

1.3 Understanding Special Conformal Transformations

Amongst the four types of transformations, two of them (translations and rotations) are already familiar, whereas dilations and SCTs are new ingredients. Dilations are a bit intuitive, but the nature of SCTs are (at least to me) somewhat unintuitive just looking at the table above. In order to understand these, we first start by defining inversion in the sphere:

$$x^\mu \mapsto \hat{\mathcal{I}}[x^\mu] := \frac{x^\mu}{x\cdot x} \tag{1.7}$$

(This is the same sphere inversion that we're familiar with from having suffered through electromagnetism.) Since sphere inversion is a map based only on the input point values and not a continuous parameter, it's a discrete transformation like parity ($\hat{\mathcal{P}}[(t,\vec{r})] := (t,-\vec{r})$) and time reversal ($\hat{\mathcal{T}}[(t,\vec{r})] := (-t,\vec{r})$), as opposed to the continuous transformations we just tabled above. Stated slightly more formally, while

$\hat{\mathcal{J}}$ is in $\mathsf{O}(d)$, it's not in $\mathsf{SO}(d)$, and is thus also not in the connected part of the conformal group. This is why this didn't show up when we considered (1.6a): since it's not in the connected part of the conformal group, it doesn't show up by exponentiating a Lie algebra element (or, equivalently, looking at the set of infinitesimal transformations, which are generated by exponentiating the Lie algebra elements).

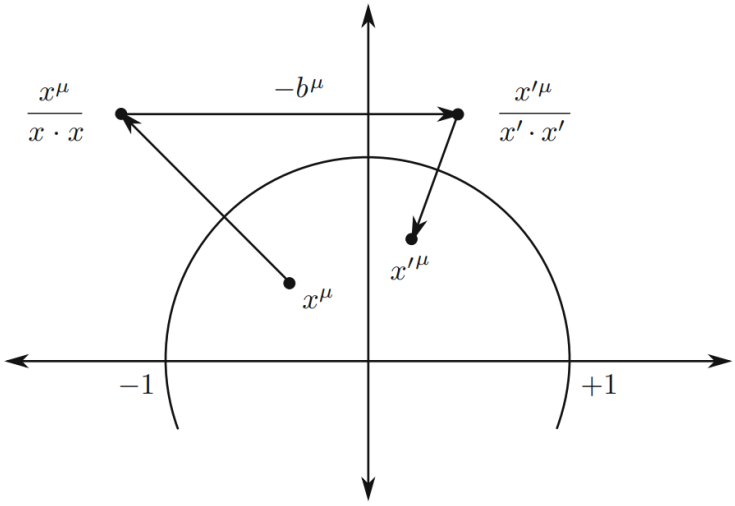

Figure 2: A sample SCT, given by the composition of a sphere inversion (the first arrow), a translation outside the sphere (the second arrow), and a second sphere inversion (the third arrow). Taken from [1].

Using (1.7), we can express SCTs as a sequence of a sphere inversion, a translation (*outside* the sphere), and another sphere inversion. This is shown in Figure 2, taken from [1]. In terms of inversions, the expression for finite SCTs simplifies substantially. In terms of the labelling in the figure, this corresponds to the process:

$$\begin{aligned} x^\mu \mapsto \frac{x^\mu}{x\cdot x} \mapsto \frac{x'^\mu}{x'\cdot x'} &:= \frac{x^\mu}{x\cdot x} + b^\mu \mapsto x'^\mu \\ x^\mu \mapsto x'^\mu &:= \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)} \end{aligned} \tag{1.8}$$

These SCTs are a "new" type of transformation, unique to the conformal group. For SCTs, the scale factor $\Lambda(x)$ for $g_{\mu\nu} \mapsto \Lambda(x)\, g_{\mu\nu}$ is given by:

$$\Lambda(x) = (1-2(b\cdot x)+(b\cdot b)(x\cdot x))^2 \tag{1.9}$$

Although we used the sphere inversion to derive a more comprehensible form of the SCTs, we note that invariance under inversion and translations isn't *quite* the same as invariance under SCTs. In particular, since $\hat{\mathcal{J}}$ isn't in the connected component of $\mathsf{SO}(d)$ (as mentioned above), invariance under $\hat{\mathcal{J}}$ is stronger condition. We mentioned earlier that inversion is a discrete transformation like parity and time reversal; in fact, parity and sphere inversion are conjugate transformations in the conformal group [8]. As such, parity-invariant conformal field theories are sphere-inversion-invariant, and vice-versa[2].

We finally note that SCTs aren't globally defined; in fact, for *every* x^μ, we can choose a specific b^μ

[2] [8] specifically refers to CFT calculations checking inversion invariance rather than SCT invariance as "pedestrian". Pretty strong language, but it certainly helps me keep a mental note not to do that.

(namely, $b^\mu = (b\cdot b)x^\mu$) such that the denominator $1-2(b\cdot x)+(b\cdot b)(x\cdot x)$ vanishes. This specific value of b^μ maps x^μ to ∞, and even if we include the numerator, L'Hôpital's rule doesn't help out. Thus, when trying to define the set of all finite conformal transformations, we need to embed our $\mathbb{R}^{d,0}$ or $\mathbb{R}^{d-1,1}$ into a compact manifold. The embedding itself must, of course, be a conformal map. The manifold must also have the overall inner product structure we're familiar with from relativity (i.e., the normal inner product as always, and one dimension having the opposite sign); these are called Lorentzian manifolds. The embedding of a not-necessarily-compact Lorentzian manifold into a compact Lorentzian manifold using a conformal embedding map is a conformal compactification.

1.4 Table of Conformal Transformations

Grouping everything in §1.2 and §1.3 together, indicating the number of each transformations in a system with signature (p, q, $r=0$) (as in (1.1)), and indicating the scale factor of the transformation using (1.1), we have:

Type	Finite Version	Generator
Translations	$x^\mu \mapsto x^\mu + a^\mu$	$\hat{P}_\mu = -\mathrm{i}\hat{\partial}_\mu$
Dilations	$x^\mu \mapsto \alpha x^\mu$	$\hat{D} = -\mathrm{i}x^\mu\hat{\partial}_\mu$
Rotations	$x^\mu \mapsto m^\mu_\nu x^\nu$	$\hat{L}_{\mu\nu} = \mathrm{i}\big(x_\mu\hat{\partial}_\nu - x_\nu\hat{\partial}_\mu\big)$
Special Conformal Transformations	$x^\mu \mapsto \dfrac{x^\mu-(x\cdot x)}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)b^\mu}$	$\hat{K}_\mu = -\mathrm{i}\big(2x_\mu x^\nu\hat{\partial}_\nu - (x\cdot x)\hat{\partial}_\mu\big)$

Type	Number	Scale Factor (Λ)
Translations	$(p+q)=d$	1
Dilations	1	$\dfrac{1}{\alpha^2}$
Rotations	$\dfrac{(p+q+1)(p+q-1)}{2} = \dfrac{(d+1)(d-1)}{2}$	1
Special Conformal Transformations	$(p+q)=d$	$(1-2(b\cdot b)+(b\cdot b)(x\cdot x))^2$

1.5 The Conformal Algebra in $d \geq 3$

As always, by taking the commutators of everything with everything else, we derive the algebra, which is done the same way as in Homework 2. This gives the algebra as:

$$\begin{aligned}
\big[\hat{D},\hat{P}_\mu\big] &= \mathrm{i}\hat{P}_\mu \\
\big[\hat{D},\hat{K}_\mu\big] &= -\mathrm{i}\hat{K}_\mu \\
\big[\hat{K}_\mu,\hat{P}_\nu\big] &= 2\mathrm{i}\big(g_{\mu\nu}\hat{D} - \hat{L}_{\mu\nu}\big) \\
\big[\hat{K}_\rho,\hat{L}_{\mu\nu}\big] &= \mathrm{i}\big(g_{\rho\mu}\hat{K}_\nu - g_{\rho\nu}\hat{K}_\mu\big) \\
\big[\hat{P}_\rho,\hat{L}_{\mu\nu}\big] &= \mathrm{i}\big(g_{\rho\mu}\hat{P}_\nu - g_{\rho\nu}\hat{P}_\mu\big) \\
\big[\hat{L}_{\mu\nu},\hat{L}_{\rho\sigma}\big] &= \mathrm{i}\big(g_{\nu\rho}\hat{L}_{\mu\sigma} + g_{\mu\sigma}\hat{L}_{\nu\rho} - g_{\mu\rho}\hat{L}_{\nu\sigma} - g_{\nu\sigma}\hat{L}_{\mu\rho}\big)
\end{aligned} \tag{1.10}$$

Unsurprisingly, if we get rid of $\hat{D}$ and $\hat{K}_\mu$, this reduces to the Poincaré algebra. For $d \geq 3$, we have d translations, d SCTs, $d(d-1)/2$ rotations, and 1 dilation. Overall, this gives $(d+2)(d+1)/2$ generators, which is exactly the number of generators for an $\mathfrak{so}(d+2)$-type algebra. This isn't a coincidence: we can define the alternate generators $\hat{J}_{mn}$ by:

$$\hat{J}_{\mu\nu} = \hat{L}_{\mu\nu}; \quad \hat{J}_{-1\mu} = \frac{\hat{P}_\mu - \hat{K}_\mu}{2}; \quad \hat{J}_{-10} = \hat{D}; \quad \hat{J}_{0\mu} = \frac{\hat{P}_\mu + \hat{K}_\mu}{2} \tag{1.11}$$

This gives $\hat{J}_{mn}$ with $m, n = -1, 0, 1, \cdots, d-1$ satisfying the commutation relations:

$$\left[\hat{J}_{mn}, \hat{J}_{rs}\right] = \mathrm{i}\left(g_{ms}\,\hat{J}_{nr} + g_{nr}\,\hat{J}_{ms} - g_{mr}\,\hat{J}_{ns} - g_{ns}\,\hat{J}_{mr}\right) \tag{1.12}$$

- For d-dimensional Euclidean space $\mathbb{R}^{d,0}$, $g_{mn} = \mathsf{diag}(-1, 1, \cdots, 1)$. Then, $\left[\hat{J}_{mn}, \hat{J}_{rs}\right]$ is the commutation relation for $\mathfrak{so}(d+1, 1)$.
- For the d-dimensional Minkowski space $\mathbb{R}^{d-1,1}$, $g_{mn} = \mathsf{diag}(-1, -1, 1, \cdots, 1)$. Then, $\left[\hat{J}_{mn}, \hat{J}_{rs}\right]$ is the commutation relation for $\mathfrak{so}(d, 2)$.

Thus, in general for $d = p + q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is $\mathsf{SO}(p+1, q+1)$.

1.6 Matrix Representations of the Conformal Algebra in $d \geq 3$

Now, we consider the matrix representations of the algebra acting on fields. For the Poincaré subgroup, these expressions are already familiar from perturbative QFT. In particular, for a field $\mathfrak{F}(x=0)$, the matrix representation of $\hat{L}_{\mu\nu}$ acting on $\mathfrak{F}(0)$ is given by:

$$\hat{L}_{\mu\nu}\,\mathfrak{F}(0) = S_{\mu\nu}\,\mathfrak{F}(0) \tag{1.13}$$

Here, $S_{\mu\nu}$ is the spin operator corresponding to the spin structure of $\mathfrak{F}$. Similarly, we can denote the matrix representations of $\hat{D}$ and $\hat{K}_\mu$ at the origin by:

$$\hat{D}\,\mathfrak{F}(0) = \tilde{\Delta}\,\mathfrak{F}(0) \tag{1.14}$$

$$\hat{K}_\mu\,\mathfrak{F}(0) = \kappa_\mu\,\mathfrak{F}(0) \tag{1.15}$$

($S_{\mu\nu}$, $\tilde{\Delta}$, and κ_μ continue to follow the commutation relations with each other given in (1.10).) Translating $\hat{L}_{\mu\nu}$ away from the origin via $\mathrm{e}^{\mathrm{i}x^\sigma \hat{P}_\sigma}\,\hat{L}_{\mu\nu}\,\mathrm{e}^{-\mathrm{i}x^\sigma \hat{P}_\sigma}$ and using the Baker-Campbell-Hausdorff expansion, we have:

$$\mathrm{e}^{\mathrm{i}x^\sigma \hat{P}_\sigma}\,\hat{L}_{\mu\nu}\,\mathrm{e}^{-\mathrm{i}x^\sigma \hat{P}_\sigma} = \hat{L}_{\mu\nu} - x_\mu \hat{P}_\nu + x_\nu \hat{P}_\mu \tag{1.16}$$

Similarly, we can translate the dilation and SCT operators away from the origin:

$$\mathrm{e}^{\mathrm{i}x^\sigma \hat{P}_\sigma}\,\hat{D}\,\mathrm{e}^{-\mathrm{i}x^\sigma \hat{P}_\sigma} = \hat{D} + x^\sigma \hat{P}_\sigma \tag{1.17}$$

$$\mathrm{e}^{\mathrm{i}x^\sigma \hat{P}_\sigma}\,\hat{K}_\mu\,\mathrm{e}^{-\mathrm{i}x^\sigma \hat{P}_\sigma} = \hat{K}_\mu + 2x_\mu \hat{D} - 2x^\nu \hat{L}_{\mu\nu} + 2x_\mu\left(x^\nu \hat{P}_\nu\right) - x^2 \hat{P}_\mu \tag{1.18}$$

From (1.16), we can derive the expected relations for the actions of the Poincaré elements:

$$\hat{P}_\mu \mathfrak{Ş}(x) = -\mathrm{i}\partial_\mu \mathfrak{Ş}(x) \tag{1.19}$$

$$\hat{L}_{\mu\nu} \mathfrak{Ş}(x) = \mathrm{i}\big(x_\mu \partial_\nu - x_\nu \partial_\mu\big)\mathfrak{Ş}(x) + S_{\mu\nu}\, \mathfrak{Ş}(x) \tag{1.20}$$

The same way, we derive the relation for the actions of the dilation and SCT operators:

$$\hat{D}\, \mathfrak{Ş}(x) = \big(-\mathrm{i}x^\sigma \partial_\sigma + \tilde{\Delta}\big)\mathfrak{Ş}(x) \tag{1.21}$$

$$\hat{K}_\mu \mathfrak{Ş}(x) = \big(\kappa_\mu + 2x_\mu \tilde{\Delta} - x^\sigma S_{\mu\sigma} - 2\mathrm{i}x_\mu x^\sigma \partial_\sigma + \mathrm{i}x^2 \partial_\mu\big)\mathfrak{Ş}(x) \tag{1.22}$$

From (1.10), we see that $\tilde{\Delta}$ commutes with $S_{\mu\nu}$; from Schur's Lemma, this tells us that $\tilde{\Delta}$ is a multiple of the identity (i.e., a c-number). Since dilations correspond to a noncompact transformation, this is a nonunitary operator, and so we have $\tilde{\Delta} = -\mathrm{i}\Delta$, where Δ is the scaling dimension of $\mathfrak{Ş}(x)$:

$$\mathfrak{Ş}(\lambda x) = \lambda^{-\Delta}\, \mathfrak{Ş}(x) \tag{1.23}$$

From (1.14), we see that the scaling dimension is an eigenvalue of the dilation operator (which is maybe unsurprising, given what the dilation operator does). The relationship $\tilde{\Delta} = -\mathrm{i}\Delta\mathbb{1}$ further gives the property that the matrices κ_μ vanish. Applying these two relations to (1.19)–(1.22) gives us the transformation rules for $\mathfrak{Ş}$ under the action of the conformal algebra. (In the interests of space, I'll simply skip rewriting this.)

2 Correlator Expressions from Conformal Invariance

From conformal symmetry alone, without even writing down a Lagrangian, it already is possible to derive important properties of *any* CFT. In particular, we can constrain several properties of n-point functions, as well as properties of the dimensions of operators; here, we look at the forms of 2-, 3-, and 4-point correlation functions. In principle, this can be expanded to correlation functions involving more operators; however, it turns out that higher-order terms can be reduced to products of these, as will be discussed in §6.1.

2.1 General Two-Point Correlator and the Definitions of Primaries and Descendants

The conformal algebra (1.10) and its matrix representations (1.13)–(1.15) (along with the relations $\tilde{\Delta} = -\mathrm{i}\Delta\mathbb{1}$ and $\kappa_\mu = 0$) are *already* enough to substantially constrain the form of expectation values. In particular, from (1.21) and from rotation and translation invariance, we require that the two-point function of two *generic* operators $\hat{\text{ओ}}_i$, $\hat{\text{ओ}}_j$ depend only on the magnitude of the distance between them:

$$\big\langle \hat{\text{ओ}}_i\, \hat{\text{ओ}}_j \big\rangle = \Big\langle 0 \Big| \mathcal{T}\Big\{ \hat{\text{ओ}}_i(x_i)\, \hat{\text{ओ}}_j(x_j) \Big\} \Big| 0 \Big\rangle = f\big(|x_i - x_j|\big) \tag{2.1}$$

If we further impose that the vacuum be scale invariant (which seems like an eminently reasonable

assumption), we have:

$$\hat{D}\,|0\rangle = 0 \tag{2.2}$$

From this, we have $\left\langle 0\middle|\left[\hat{D}, \hat{\text{ओ}}\right]\middle|0\right\rangle = 0$ for *any* operator $\hat{\text{ओ}}$. This, of course, includes the product $\hat{\text{ओ}}_i\,\hat{\text{ओ}}_j$. Expanding out the expectation value of the commutator $\left[\hat{D}, \hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\right]$, we have:

$$\begin{gathered}\left\langle 0\middle|\left[\hat{D}, \hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\right]\middle|0\right\rangle = \left\langle 0\middle|\left[\hat{D}, \hat{\text{ओ}}_i\right]\hat{\text{ओ}}_j + \hat{\text{ओ}}_i\left[\hat{D}, \hat{\text{ओ}}_j\right]\middle|0\right\rangle = 0 \\ \left(x_i^\mu \partial_{\mu i} + \Delta_i + x_j^\mu \partial_{\mu j} + \Delta_j\right)\left\langle 0\middle|\hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\middle|0\right\rangle = 0\end{gathered} \tag{2.3}$$

The solution to this gives a power law expression, up to a multiplicative constant $\hbar \in \mathbb{R}$:

$$\left\langle \hat{\text{ओ}}_i\,\hat{\text{ओ}}_j \right\rangle = \frac{\hbar}{\left|x_i - x_j\right|^{\Delta_i + \Delta_j}} \tag{2.4}$$

In addition to this relationship, derived *purely* from the action of $\hat{D}$ on generic operators, we also note that $\hat{K}_\mu$ is a sort of lowering operator for dimension:

$$\hat{D}\left\{\hat{K}_\mu \hat{\text{ओ}}\right\} = \left(\left[\hat{D}, \hat{K}_\mu\right] + \hat{K}_\mu \hat{D}\right)\hat{\text{ओ}} = \left(-\mathrm{i}\hat{K}_\mu + \mathrm{i}\Delta\hat{K}_\mu\right)\hat{\text{ओ}} = \mathrm{i}(\Delta - 1)\hat{K}_\mu \hat{\text{ओ}} \tag{2.5}$$

As a result, there must be some operator $\hat{\text{पे}}$ such that $\hat{K}_\mu$ lowers $\hat{\text{पे}}$ to zero:

$$\hat{K}_\mu \hat{\text{पे}} = 0 \tag{2.6}$$

This is our first definition of a primary operator of dimension Δ. A different definition involving the Jacobian (1.2) is that $\hat{\text{पे}}$, with spin corresponding to the irreducible representation (irrep) $\mathcal{R}$ of $\mathsf{SO}(d)$ (as before, implemented by $\rho_b^a \in \mathcal{R}\{\mathsf{SO}(d)\}$), transforms for *local* conformal transformations as:

$$\hat{\text{पे}}(x) \mapsto \frac{\rho_b^a\left((m_\nu^\mu)^{-1}\right)}{\Lambda^{\Delta/2}}\,\hat{\text{पे}}(x') \tag{2.7}$$

(Here, as before, $m_\nu^\mu \in \mathsf{SO}(d)$.) In the case that (2.7) is only satisfied for *global* conformal transformations, $\hat{\text{पे}}$ is defined as a quasi-primary operator. (Operators that are neither primary nor quasi-primary are called secondary operators.)

Analogously to how the SCT operator $\hat{K}_\mu$ acts as a lowering operator for dimension, the momentum operator $\hat{P}_\mu$ acts as a raising operator for dimension:

$$\hat{D}\left\{\hat{P}_\mu \hat{\text{ओ}}\right\} = \left(\left[\hat{D}, \hat{P}_\mu\right] + \hat{P}_\mu \hat{D}\right)\hat{\text{ओ}} = \left(\mathrm{i}\hat{P}_\mu + \mathrm{i}\Delta\hat{P}_\mu\right)\hat{\text{ओ}} = \mathrm{i}(\Delta + 1)\hat{P}_\mu \hat{\text{ओ}} \tag{2.8}$$

From the raising operator properties of the momentum operator, we can operate on a primary operator $\hat{\text{पे}}$ with $\hat{P}_\mu$ (ad infinitum) to get the level $N \geq 1$ descendant operators $\hat{P}_{\mu_1} \cdots \hat{P}_{\mu_N}\hat{\text{पे}}$ of dimension $\Delta + N$.

The set of $\hat{\mathfrak{q}}$ along with all of its descendants is defined as a conformal family:

$$\mathcal{C}(\mathfrak{q}) := \left\{\hat{\mathfrak{q}},\, \hat{P}_{\mu_1}\hat{\mathfrak{q}},\, \cdots,\, \hat{P}_{\mu_1}\cdots\hat{P}_{\mu_N}\hat{\mathfrak{q}}\right\} \tag{2.9}$$

This is reminiscent of how we constructed irreps of $\mathsf{SU}(2)$ from the highest-weight state; here, the only difference is that we're constructing representations of the conformal group (obviously) and that we're starting instead from the *lowest*-weight state. The fact that $\hat{P}_\mu$ and $\hat{K}_\mu$ act as raising and lowering operators for dimension gives us additional structure, which we discuss in §3.

2.2 Primary Field Two-Point Correlators

Primary fields and operators have stronger constraints than general operators, which similarly even more strongly constrain the form of expectation values. (In deriving these, we follow [2–4, 9, 12].) For a unitary transformation $\hat{U}$ generated by a transformation, we require the n-point function of the product of n primary operators satisfy:

$$\begin{aligned}\left\langle\hat{\mathfrak{q}}(x_1)\cdots\hat{\mathfrak{q}}(x_n)\right\rangle &= \left\langle\left(\hat{U}\,\hat{\mathfrak{q}}(x_1)\,\hat{U}^{-1}\right)\cdots\left(\hat{U}\,\hat{\mathfrak{q}}(x_n)\,\hat{U}^{-1}\right)\right\rangle \\ &= \mathcal{R}_1\left\{\left(m_1{}^{\mu}_{\nu}\right)^{-1}\right\}\left(\Lambda(x_1')\right)^{\Delta_1}\cdots\mathcal{R}_n\left\{\left(m_n{}^{\mu}_{\nu}\right)^{-1}\right\}\left(\Lambda(x_1')\right)^{\Delta_n}\left\langle\hat{\mathfrak{q}}(x_1')\cdots\hat{\mathfrak{q}}(x_n')\right\rangle\end{aligned} \tag{2.10}$$

In addition to this, for conformal transformations specifically, we have the relationship:

$$\left(x_i - x_j\right)^2 = \frac{\left(x_i' - x_j'\right)^2}{\Lambda(x_i')\Lambda\left(x_j'\right)} \tag{2.11}$$

Applying (2.10) and (2.11) to (2.4) (without the spin representation factors), we have the two-point function of scalar (spinless) primaries as:

$$\left\langle\hat{\mathfrak{q}}_i(x_i)\,\hat{\mathfrak{q}}_j(x_j)\right\rangle = \frac{\hbar\,\delta_{\Delta_i\Delta_j}}{\left|x_i - x_j\right|^{2\Delta}} \tag{2.12}$$

The two-point function for primary operators with spin is slightly more complicated to derive; here, we follow [3, 4]. By relating (2.10) to (2.4) (*with* the spin representation factors), we can extract a general expression for the two-point function in terms of a two-point function for two arbitrary but fixed points. From this, examining the stabiliser group of $\mathsf{SO}(d+1,1)$ for those specific points (i.e., the subgroup of $\mathsf{SO}(d+1,1)$ that leaves two specific points invariant), and further examining the requirement of invariance under the sphere inversion $\hat{\mathcal{J}}$ defined in (1.7), gives us an expression for the two-point function. For the spin-s_i and spin-s_j primary operators $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_{s_i}}$ and $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_{s_j}}$ respectively (in the symmetric traceless representation), this procedure gives:

$$\left\langle\hat{\mathfrak{q}}_i^{\mu_1\cdots\mu_{s_i}}(x_i)\,\hat{\mathfrak{q}}_j^{\mu_1\cdots\mu_{s_j}}(x_j)\right\rangle = \frac{\hbar\,\delta_{\Delta_i\Delta_j}\,\delta_{s_i s_j}}{\left|x_i - x_j\right|^{2(\Delta-s)}}\left(\partial_{(\nu_1}\hat{\mathcal{J}}^{(\mu_1}\cdots\partial_{\nu_s)}\hat{\mathcal{J}}^{\mu_s)} - \prod_{n,\,m=1}^{s}\delta_{\mu_n\cdots\mu_m}\right) \tag{2.13}$$

(Here, the parentheses around the indices represent symmetrisation over those indices. Also, as always, we leave the full derivations for the unabridged notes.)

2.3 Primary Field Three-Point and Four-Point Correlators

Meanwhile, from Poincaré and dilation invariance, the three point function of spinless primary operators must have the general form:

$$\left\langle \hat{\mathfrak{q}}_i(x_i)\,\hat{\mathfrak{q}}_j(x_j)\,\hat{\mathfrak{q}}_k(x_k)\right\rangle = \sum_{\substack{a,\,b,\,c\\ a+b+c=\Delta_i+\Delta_j+\Delta_k}} \frac{f_{abc}}{|x_i-x_j|^a\,|x_j-x_k|^b\,|x_k-x_i|^c} \tag{2.14}$$

(Here, $\{f_{abc}\}\in\mathbb{R}$.) Imposing invariance under SCTs, using the same series of steps as for (2.12), further constrains a, b, and c:

$$a=\Delta_i+\Delta_j-\Delta_k,\quad b=\Delta_j+\Delta_k-\Delta_i,\quad c=\Delta_k+\Delta_i-\Delta_j \tag{2.15}$$

Applying this to (2.14) gives the final form of the three-point correlator of spinless primary operators as:

$$\left\langle \hat{\mathfrak{q}}_i(x_i)\,\hat{\mathfrak{q}}_j(x_j)\,\hat{\mathfrak{q}}_k(x_k)\right\rangle = \frac{f_{ijk}}{|x_i-x_j|^{\Delta_i+\Delta_j-\Delta_k}\,|x_j-x_k|^{\Delta_j+\Delta_k-\Delta_i}\,|x_k-x_i|^{\Delta_k+\Delta_i-\Delta_j}} \tag{2.16}$$

(As before, $\{f_{ijk}\}\in\mathbb{R}$.) To generalise this to the three-point functions of spin-s primaries, we follow [12–14][3]. Starting with a spin-s primary operator $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_s}$, we take the product of this with a set of auxiliary null polarisation vectors $\{\zeta_{\mu_i}\}$ (with $\zeta^2=0$). From these, this technique looks for Poincaré-invariant structures with fixed weight under the dilation operator $\hat{D}$ and which are invariant under sphere inversion $\hat{\mathcal{J}}$ defined in (1.7). Notably, by examining the product $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_s}\zeta_{\mu_1}\cdots\zeta_{\mu_i}$, we convert the problem of determining the three-point function of primaries with spin s and dimension Δ into a problem of determining the three-point function of *scalar* primaries of dimension $\Delta-s$ (which is already given by (2.16)), along with determining the specific functions that are Poincaré- and sphere-inversion invariant.

Using this technique, the three-point function for the spin-s_i, spin-s_j, and spin-s_k primary operators in the symmetric traceless representation $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_{s_i}}$, $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_{s_j}}$, and $\hat{\mathfrak{q}}^{\mu_1\cdots\mu_{s_k}}$ respectively is given by [12, 14]:

$$\left\langle \hat{\mathfrak{q}}_i^{\mu_1\cdots\mu_{s_i}}(x_i)\,\hat{\mathfrak{q}}_j^{\mu_1\cdots\mu_{s_j}}(x_j)\,\hat{\mathfrak{q}}_k^{\mu_1\cdots\mu_{s_k}}(x_k)\right\rangle = \frac{G_{(s_i,s_j,s_k)}(P_a,\,Q_a,\,S_a)}{|x_i-x_j|^{\delta_i+\delta_j-\delta_k}\,|x_j-x_k|^{\delta_j+\delta_k-\delta_i}\,|x_k-x_i|^{\delta_k+\delta_i-\delta_j}} \tag{2.17}$$

Here, $\delta_a=\Delta_a-s_a$, $a\in\{i,\,j,\,k\}$ (where i, j, and k correspond to the labelling of the primaries), and $G_{(s_i,s_j,s_k)}$ is a polynomial in $(P_a,\,Q_a,\,S_a)$ that is at most linear in S_a and degree-$2s_a$ homogenous in ζ_a,

[3] [3, 4, 15] provide a different approach, using the embedding formalism. However, both approaches are compatible, as mentioned in both sets of references. I personally found the approach in [12–14] more quickly / easily digestible.

and P_a, Q_a, and S_a are the Poincaré- and sphere-inversion-invariant structures given by:

$$
\begin{aligned}
P_a &= \zeta^b \frac{(x_b - x_c)^\mu \sigma_\mu}{|x_a - x_b|^2} \zeta_c \\
Q_a &= \zeta^a \frac{(x_a - x_b)^\mu \sigma_\mu}{|x_a - x_b|^2} \big((x_b - x_c)^\nu \sigma_\nu\big) \frac{(x_c - x_a)^\rho \sigma_\rho}{|x_c - x_a|^2} \zeta_a \\
S_i &= \mathrm{i} \frac{\zeta^k (x_k - x_i)^\mu \sigma_\mu (x_i - x_j)^\nu \sigma_\nu \, \zeta_j}{|x_i - x_j||x_j - x_k||x_k - x_i|} \frac{\zeta^j (x_j - x_k)^\rho \sigma_\rho \, \zeta_k}{|x_j - x_k|^2} \\
S_j &= \mathrm{i} \frac{\zeta^i (x_i - x_j)^\mu \sigma_\mu (x_j - x_k)^\nu \sigma_\nu \, \zeta_k}{|x_i - x_j||x_j - x_k||x_k - x_i|} \frac{\zeta^k (x_k - x_i)^\rho \sigma_\rho \, \zeta_i}{|x_k - x_i|^2} \\
S_k &= \mathrm{i} \frac{\zeta^j (x_j - x_k)^\mu \sigma_\mu (x_k - x_i)^\nu \sigma_\nu \, \zeta_i}{|x_i - x_j||x_j - x_k||x_k - x_i|} \frac{\zeta^i (x_i - x_j)^\rho \sigma_\rho \, \zeta_j}{|x_i - x_j|^2}
\end{aligned}
\tag{2.18}
$$

Before continuing, we note that this entire technique involves contracting the primaries with the corresponding number of null polarisation vectors. We can retrieve the original primaries via the Thomas derivative, defined in ζ-space [14–18]:

$$
D^\mu_\zeta = \left(\frac{d-2}{2}\right) \partial_{\zeta_\mu} + \zeta^\nu \partial_{\zeta_\nu} \partial_{\zeta_\mu} - \frac{\zeta^\mu \left(\partial_{\zeta_\nu} \partial^{\zeta_\nu}\right)}{2} \tag{2.19}
$$

From this, the primary $\widehat{\mathfrak{q}}_i^{\mu_1 \cdots \mu_{s_i}}(x_i)$ can be recovered from the contraction $\widehat{\mathfrak{q}}_i^{\mu_1 \cdots \mu_{s_i}}(x_i) \zeta_{\mu_1} \cdots \zeta_{\mu_s}$ by repeatedly applying the Thomas derivative:

$$
\widehat{\mathfrak{q}}_i^{\mu_1 \cdots \mu_{s_i}}(x_i) = D^{\mu_1}_\zeta \cdots D^{\mu_s}_\zeta \left[\widehat{\mathfrak{q}}_i^{\mu_1 \cdots \mu_{s_i}}(x_i) \zeta_{\mu_1} \cdots \zeta_{\mu_s}\right] \tag{2.20}
$$

For four-point functions, conformal invariance is less constraining. In general, translation and rotation invariance imposes that functions can only depend on the absolute values of distances (as in the two-point and three-point functions) and their ratios. Under SCTs, the distance separating two points transforms as:

$$
|x'_i - x'_j| = \frac{|x_i - x_j|}{\sqrt{1 - 2(b \cdot x_i) + (b \cdot b)(x \cdot x)} \sqrt{1 - 2(b \cdot x_j) + (b \cdot b)(x \cdot x)}} \tag{2.21}
$$

For two-point and three-point functions, no functions of ratios can emerge, thus constraining these to be of the forms already derived. For four-point functions, however, the cross-ratios (also known as anharmonic ratios) are also invariant:

$$
\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} =: \sqrt{u}\,; \qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|} =: \sqrt{v} \tag{2.22}
$$

In terms of the u and v defined this way, the general expression of the four-point correlator of spinless

primary operator is given by:

$$\left\langle \hat{\mathfrak{q}}_i(x_i)\,\hat{\mathfrak{q}}_j(x_j)\,\hat{\mathfrak{q}}_k(x_k)\,\hat{\mathfrak{q}}_m(x_m)\right\rangle = \prod_{i<j}\frac{g(u,v)}{\left|x_i-x_j\right|^{-(\Delta_i+\Delta_j)+\sum_{n=1}^{4}\Delta_n/3}} \tag{2.23}$$

Here, $g(u,v)$ is a generic function of u and v. (I haven't found anywhere an expression for the four-point function of operators with spin, and the discussion and result of [19] seems to imply that a closed-form expression is yet to be derived.) We note that this is invariant under permutations of the points the operators are evaluated at; this gives rise to the invariance conditions on $g(u,v)$ given by:

$$g(u,v) = g\left(\frac{u}{v},\frac{1}{v}\right), \quad g(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathfrak{q}}} g(v,u) \tag{2.24}$$

Although in principle we can use conformal invariance to constrain the form of correlation functions with higher numbers of operators, the presence of the cross-ratios means that these expressions are far less constraining, and thus far less useful. We can instead use the operator product expansion (OPE) to reduce those correlation functions to sums of one of the types of functions given in this section. The OPE, the constraints on it due to conformal invariance, and the technique of reducing correlation functions are discussed in §6.

3 Radial Quantisation

3.1 State-Operator Correspondence and Radial Quantisation

As was shown in §2.1, the momentum operator $\hat{P}_\mu$ and SCT operator $\hat{K}_\mu$ act as sort of raising and lowering operators for dimension. This strongly suggests that we can make a correspondence between our operators and states in a vector space. Making this correspondence requires a foliation [2–4] of spacetime; i.e., a choice of how we subdivide d-dimensional spacetime into $(d-1)$-dimensional regions[4]. This is something we already did all the way back in quantum mechanics: for a d-dimensional system in nonrelativistic quantum mechanics (including time in d), the Hilbert space at a given time t is our foliation. The eigenstates of the Hamiltonian defined the states in Hilbert space, and the Hamiltonian then served as the time translation operator moving between folations.

Our intuitions from quantum mechanics directly map over to the foliations of conformally invariant systems. In quantum mechanics, the eigenstates of the Hamiltonian defined the Hilbert spaces, corresponding to specific time slices, and the Hamiltonian then moved between different foliations. Similarly, in conformally invariant systems, the eigenstates of the dilation operator (i.e., operators with dimension Δ) define the space we live in, corresponding to specific time slices, and the dilation operator then moves between them.

In this foliation, we divide $\mathbb{R}^d$ using $(d-1)$-dimensional spheres S^{d-1} of different radii, all centred at the

[4] A small note after finishing this report: looking back on it, I *really* enjoyed using the word "foliation" in this section.

origin, with the metric given by:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\vec{\mathbf{n}}^2 = \mathrm{e}^{2\mathfrak{a}}\left(\mathrm{d}\mathfrak{a}^2 + \mathrm{d}\vec{\mathbf{n}}^2\right) \tag{3.1}$$

In the second equality, we define a new coordinate $\mathfrak{a} := \ln r$, which takes the place of the time coordinate in this foliation. The dilation operator then corresponds to changing the radius of the sphere, as shown in Figure 2L, taken from [3, 4]. Specifically, we define the analogue of the unitary time evolution operator by:

$$\hat{U}_{\mathfrak{a}} := \mathrm{e}^{\mathrm{i}\hat{D}\mathfrak{a}} \tag{3.2}$$

As a result, we can identify primary and descendant operators with states, labelled by dimension Δ and spin s (corresponding to the irreducible representation $\mathcal{R}$ of $\mathsf{SO}(d)$); with the action of the unitary time evolution analogue given correspondingly:

$$\hat{D}\,|\Delta, s\rangle = \mathrm{i}\Delta\,|\Delta, s\rangle\,, \quad M_{\mu\nu}\,|\Delta, s\rangle = \left(\Sigma_{\mu\nu}\right)|\Delta, s\rangle\,, \quad \hat{U}_{\mathfrak{a}} = \mathrm{e}^{\mathrm{i}\hat{D}\mathfrak{a}}\,|\Delta\rangle = \mathrm{e}^{-\Delta\mathfrak{a}}\,|\Delta\rangle = r^{-\Delta}\,|\Delta\rangle \tag{3.3}$$

This is the state-operator correspondence. (This is in contrast with states in Fock space in perturbative field theory, which correspond to specific particles of given spin, charge, flavour, and four-momentum.) From here on, we denote the Hilbert space at a given radius r as $\mathcal{V}_r$. (The reason for this notation will be clear very shortly.) In terms of the variables defined in (3.1), the radial operators are defined in terms of the original operators by:

$$\hat{\mathcal{O}}_{\text{rad}}(\mathfrak{a}, \vec{\mathbf{n}}) := \mathrm{e}^{\Delta\mathfrak{a}}\,\hat{\mathcal{O}}_{\text{init}}(x = \mathrm{e}^{\mathfrak{a}}\,\vec{\mathbf{n}}) \tag{3.4}$$

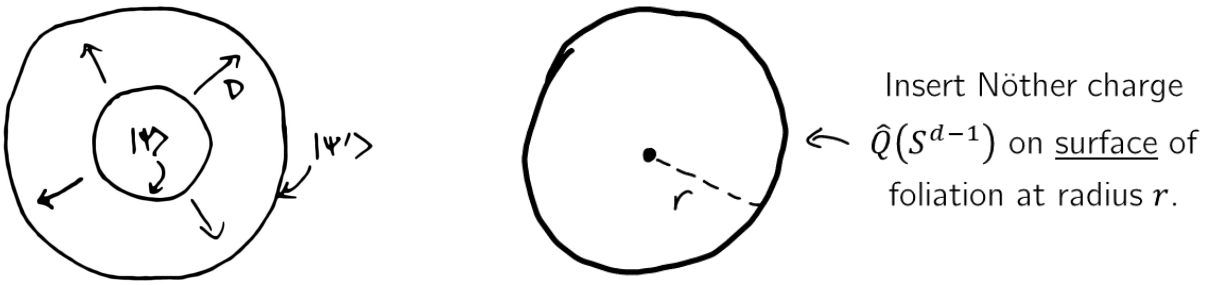

Figure 3: **Left**: A foliation of d-dimensional spacetime into spheres of dimension $(d-1)$, on which the states live. The dilation operator moves between spheres of different radii, corresponding to moving forward in time. Taken from [3, 4].

Right: Insertion of a generic Nötherian charge $\hat{Q}$ on the surface of the sphere S^{d-1} of radius r.

In addition to the eigenstates of $\hat{D}$, we saw in §2.1 that $\hat{P}_\mu$ and $\hat{K}_\mu$ served as raising and lowering operators for conformal dimension. These served as the motivation for defining the primaries as the highest-weight states via (2.6), and of the conformal family (2.9) by repeatedly applying $\hat{P}_\mu$. This directly resembled the construction of states of various spin projection in QM (the classic introductory example being the spin-half z-basis $|\pm z\rangle$). The construction of the conformal family strongly suggests that a similar construction can serve as the basis for the Hilbert space in our given foliation.

In a given theory, we may have several primary operators and thus several conformal families; however, the set of *all* conformal families taken together does indeed form the basis of the foliation [20–22]. These families taken together defines the generalised Verma module $\mathcal{V}$:

$$\mathcal{V} = \bigoplus_{\hat{\mathfrak{p}}} \mathcal{C}(\hat{\mathfrak{p}}) \tag{3.5}$$

Each individual conformal family is a specific representation of the conformal group; then, $\mathcal{V}$ as a whole is the space that the operators (and thus the states) we're considering in our theory live on. The eigenbasis of $\mathcal{V}$ is the set of *all* operators in *all* of the conformal families at a given radius r, defined via (2.9). The basis vectors for the generalised Verma module are the conformal multiplets, denoted by $\{|\mathfrak{f}_i\rangle\}$:

$$\mathcal{B}(\mathcal{V}) = \{|\mathfrak{f}_i\rangle\}; \quad \mathcal{V} = \bigoplus_{\hat{\mathfrak{p}}} \mathcal{C}(\hat{\mathfrak{p}}) = \mathrm{span}(\{|\mathfrak{f}_i\rangle\}) \tag{3.6}$$

In our theory, we may have several different primaries; however, we can use the definitions (3.5) and (3.6) to express $\{|\mathfrak{f}_i\rangle\}$ explicitly. For a given primary operator $\hat{\mathfrak{p}}$, we can define $\hat{\Pi}_{\mathfrak{p}}$ as the projector onto the conformal family of that specific $\hat{\mathfrak{p}}$, with the sum of all such projectors summing to $\mathbb{1}$:

$$\hat{\Pi}_{\mathfrak{p}} := \sum_{i,j} \left|P_{\mu_i}\mathfrak{p}\right\rangle \left\langle P_{\mu_i}\mathfrak{p}\middle|P_{\mu_j}\mathfrak{p}\right\rangle \left\langle P_{\mu_j}\mathfrak{p}\right|, \quad \mathbb{1} = \sum_{\mathfrak{p}} \hat{\Pi}_{\mathfrak{p}} = \sum_{\mathfrak{p}} \sum_{i,j} \left|P_{\mu_i}\mathfrak{p}\right\rangle \left\langle P_{\mu_i}\mathfrak{p}\middle|P_{\mu_j}\mathfrak{p}\right\rangle \left\langle P_{\mu_j}\mathfrak{p}\right| \tag{3.7}$$

From the basis $\{|\mathfrak{f}_i\rangle\}$ given in (3.6), our intuitions from QM follow as normal. For instance, an insertion of the identity is given by the path integral taken over the fields $\mathfrak{f}_i\big((\partial B)_r\big)$ at the *surface* ∂B of the sphere at radius r, and the eigendecomposition of a generic operator $\hat{\mathfrak{o}}$ given correspondingly:

$$\mathbb{1} = \int_{\mathfrak{f}_i((\partial B)_r)} \mathcal{D}\mathfrak{f}_i \ |\mathfrak{f}_i\rangle\langle\mathfrak{f}_i|, \quad |\mathfrak{o}\rangle = \int_{\mathfrak{f}_i((\partial B)_r)} \mathcal{D}\mathfrak{f}_i \ |\mathfrak{f}_i\rangle \langle\mathfrak{f}_i|\mathfrak{o}\rangle \tag{3.8}$$

As in QM, where we express a given state in terms of the spectrum and use that expression to perform calculations, (3.8) allows us to perform calculations on generic states $|\mathfrak{o}\rangle$ by decomposing them into the spectrum we have on hand for our theory. As an example, a generic operator $|\mathfrak{o}\rangle$ at a point y away from the radius r that we're living on *automatically* decomposes into a linear combination of $|\mathfrak{f}_i\rangle$s:

$$|\mathfrak{o}(y)\rangle = \hat{\mathfrak{o}}\,|0\rangle = \mathrm{e}^{(y-r)^\mu \hat{P}_\mu}\, \hat{\mathfrak{o}}(y)\, \mathrm{e}^{-(y-r)^\mu \hat{P}_\mu} = \mathrm{e}^{(y-r)^\mu \hat{P}_\mu}\ |\mathfrak{o}\rangle = \sum_{n=0}^{\infty} \frac{\left((y-r)^\mu \hat{P}_\mu\right)^n}{n!} |\mathfrak{o}\rangle = \sum_i c_i\, |\mathfrak{f}_i\rangle \tag{3.9}$$

In the last expression, we used (2.9) and (3.5)–(3.6) to identify this sum as the sum over the generalised Verma module, the basis for which we already defined by $|\mathfrak{f}_i\rangle$; the c_is are just the complex coefficients.

Analogously to (3.8), we can insert a Nötherian charge $\hat{Q}_\epsilon$ onto the surface $(\partial B)_r$, depicted in Figure 2R. Here, $\hat{Q}_\epsilon$ generates the infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$. Combining the individual transforma-

tions (1.19)–(1.22) and inserting this on the sphere of radius r with eigenbasis $\{|\text{फ}_i\rangle\}$, we have:

$$\hat{Q}_\epsilon\left(\sum_i c_i\,|\text{फ}_i\rangle\right) = \sum_i c_i \int\limits_{\text{फ}_i((\partial B)_r)} \mathcal{D}\text{फ}_i\left(\epsilon\cdot\partial + \frac{\Delta(\partial\cdot\epsilon)}{d} - \frac{(\partial^\mu\epsilon^\nu)S_{\mu\nu}}{2}\right)|\text{फ}_i\rangle \tag{3.10}$$

The vacuum state $|0\rangle$ is given by taking the path integral over the *interior* of the sphere at radius $\text{२} = r$, with no operator insertions and with the boundary condition $\Delta(\text{२},\vec{\text{n}}) = \Delta(r,\vec{\text{n}})$, as shown in Figure 3L, taken from [3, 4]. The coefficients of this in terms of the eigenbasis decomposition at radius r are given by:

$$\langle\text{फ}_i|0\rangle = \int\limits_{\text{फ}_i((\partial B)_\text{२}),\,\text{२}\le r} \mathcal{D}\text{फ}_i\; \text{e}^{-S[\text{फ}_i]} \tag{3.11}$$

Generic operators $\hat{\text{ओ}}$ applied to the vacuum (for instance, when constructing the asymptotic in-state) then correspond to inserting $\hat{\text{ओ}}$ into the path integral taken over the interior of the sphere, as shown in Figure 3R. The coefficients of this in terms of the eigenbasis decomposition at radius r are given by:

$$\left\langle\text{फ}_i\middle|\hat{\text{ओ}}\middle|0\right\rangle = \int\limits_{\text{फ}_i((\partial B)_\text{२}),\,\text{२}\le r} \mathcal{D}\text{फ}_i\;\hat{\text{ओ}}\;\text{e}^{-S[\text{फ}_i]} \tag{3.12}$$

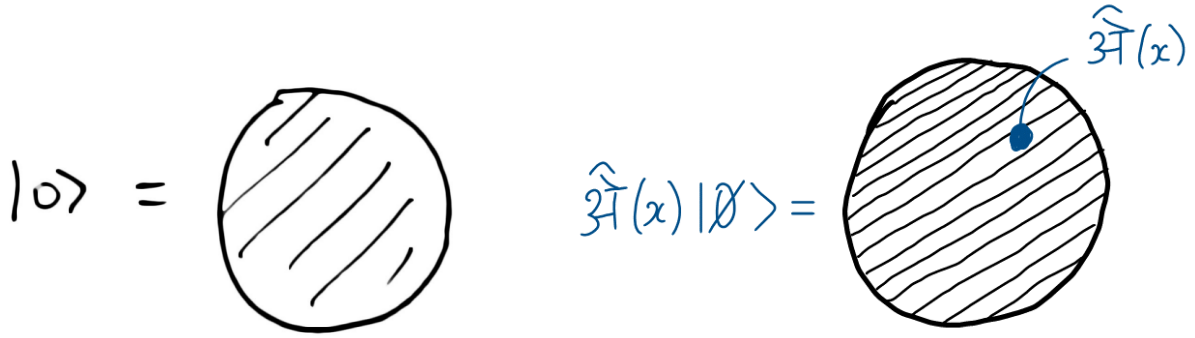

Figure 4: **Left**: The vacuum corresponds to taking the path integral of the *interior* of the sphere at radius r, with no operator insertions. Taken from [3, 4].

Right: The state $\hat{\text{ओ}}\,|0\rangle$ corresponding to the application of the operator $\hat{\text{ओ}}$ to $|0\rangle$ is given by inserting $\hat{\text{ओ}}$ into the interior of the sphere at radius r and taking the path integral.

The vacuum is invariant under the symmetries of the theory (obviously), so applying Nötherian charges $\hat{Q}$ to the vacuum gives zero. This corresponds to inserting $\hat{Q}$ either on the interior of the sphere (as in (3.10)) or in its interior (as in (3.12)), as depicted in Figure 5. We can see that these both vanish, since the charge doesn't change the topology of the surface of the sphere, and we can shrink this surface to zero inside the sphere.

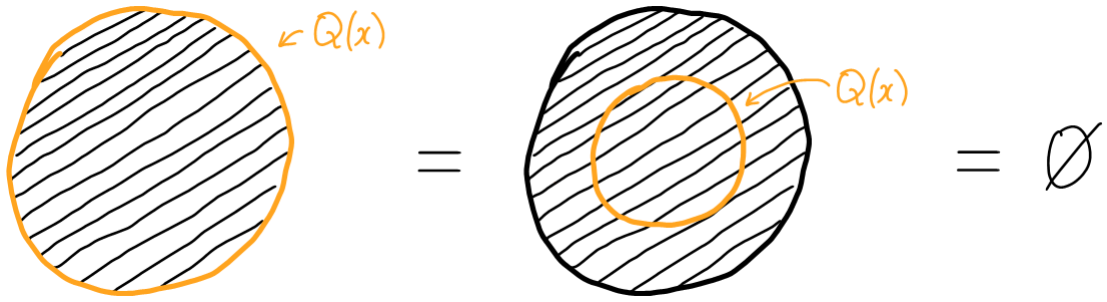

Figure 5: The vacuum is invariant under symmetry transformations, and thus applying Nöther charges $\hat{Q}$ either inside the sphere on on the surface of the sphere gives 0.

3.2 BPZ Conjugation and Radial-Ordered Products

In order to define an inner product, we must also carefully define the duals. Here, we follow [3, 4]. Since we have foliated spacetime differently from quantum mechanics and perturbative QFT, Hermitian conjugation at any arbitrary point in $\mathbb{R} \times S^{d-1}$ is not guaranteed to act in precisely the same way as it would in Hilbert or Fock space. However, we *can* define a Hermitian conjugation operation at the radial origin to match with our previous experience with Hermitian conjugation.

As such, we *start* with $\hat{\text{ओ}}^\dagger(0)$ defined at the origin in such a way that $\hat{\text{ओ}}^\dagger(0)$ *automatically* corresponds to the Hermitian conjugation action we're familiar with from QM and QFT. Our goal is, of course, to define Hermitian conjugation for arbitrary radii such that Hermitian conjugation works the way we're familiar with; the conjugate $\hat{\text{ओ}}^\dagger(0)$ at the origin will serve as the reference point operator in order to define conjugation in the way that corresponds to our previous intuitions[5]. Following [3, 4], we can then define a "non-radial Hermitian conjugate" (NRHC) at arbitrary radii, labelled $\hat{\text{ओ}}^{\dagger_{\text{NR}}}$:

$$\hat{\text{ओ}}^{\dagger_{\text{NR}}}(x) := \mathrm{e}^{x\cdot\hat{P}}\,\hat{\text{ओ}}^\dagger(0)\,\mathrm{e}^{-x\cdot\hat{P}} \tag{3.13}$$

In terms of the NRHC, we can always find a conformally-invariant two-point function $\left\langle \hat{\text{ओ}}^{a\dagger_{\text{NR}}}(x_2)\,\hat{\text{ओ}}^b(x_1)\right\rangle$ of $\hat{\text{ओ}}$ with itself at two different points in spacetime x_1 and x_2[6]. We can express this (for *any* operator, not just primaries), as:

$$\left\langle \hat{\text{ओ}}^{a\dagger_{\text{NR}}}(x_2)\,\hat{\text{ओ}}^b(x_1)\right\rangle = \frac{c_{\hat{\text{ओ}}}\, j^{ab}(|x_1-x_2|)}{|x_1-x_2|^{2\Delta_{\hat{\text{ओ}}}}} \tag{3.14}$$

(Here, a and b are spin indices, with $c_{\hat{\text{ओ}}} \in \mathbb{R}^+$ as a positive constant that depends on each operator.) From this, we denote the inverse tensor of $j^{ab}(x)$ as $j_{ba}(x) = \left(j^{ab}(x)\right)^*$, along with its normalisation:

$$j_{ba}(x) = \left(j^{ab}(x)\right)^*, \quad \sum_c \left(j^{ca}(x)\right)^* j^{cb}(x) = \delta^b_a \tag{3.15}$$

[5] I'm especially grateful to Akshay for this clarification in particular. Any mischaracterisation here is due to my mangling the explanation (although hopefully not my misunderstanding what's going on).

[6] Quoting [3, 4] directly (since I can't express it any more succinctly), "This must be the case because $\langle 0|\hat{\text{ओ}}^\dagger\hat{\text{ओ}}|0\rangle$ is the norm of a state, so it had better be zero."

Then, in terms of j^{ab}, we define the Hermitian conjugate of $\hat{\text{ओ}}$ by:

$$\hat{\text{ओ}}^{a\dagger_{\text{BPZ}}}(x) = \frac{j_{ab}(x)}{x^{2\Delta_{\hat{\text{ओ}}}}} \hat{\text{ओ}}^{b\dagger_{\text{NR}}}\left(\frac{x}{x^2}\right) \tag{3.16}$$

This is referred to as Belavin-Polyakov-Zamolodchikov (BPZ) conjugation [23], and the subscript BPZ will be kept for the rest of this paper. Using the BPZ conjugation, we can finally define the dual state $\langle \text{ओ}_a | \in \mathcal{V}_r^\star$ to the state $|\text{ओ}^a\rangle \in \mathcal{V}_r$ (where $\mathcal{V}_r$ is the generalised Verma module at the radius r, and $\mathcal{V}_r^\star$ is the dual space to $\mathcal{V}_r$):

$$\begin{aligned} \langle \text{ओ}_a | &= \left(\hat{\text{ओ}}_a(0)\,|0\rangle\right)^{\dagger_{\text{BPZ}}} = \lim_{x\to 0}\left(\hat{\text{ओ}}^a(x)\,|0\rangle\right)^{\dagger_{\text{BPZ}}} \\ &= \lim_{x\to 0}\langle 0|\,\hat{\text{ओ}}^{a\dagger_{\text{BPZ}}}(x) = \lim_{y\to\infty}\left(\frac{x}{x^2}\right)^{2\Delta} j_{ab}\left(\frac{x}{x^2}\right)\,\langle 0|\,\hat{\text{ओ}}^{b\dagger_{\text{NR}}}\left(\frac{x}{x^2}\right) \end{aligned} \tag{3.17}$$

(Here, $\langle \text{ओ}_a|$ has a lowered index, since it transforms according to the *dual* $\mathsf{SO}(d)$ representation of $|\text{ओ}^a\rangle$.) Defining the states and their duals in this way *does*, finally, give us a foliation of spacetime that allows us to carry over our understanding from quantum mechanics and perturbative QFT. For instance, we have the inner product $\langle \text{ओ}_a|\text{ओ}^a\rangle$ given by:

$$\begin{aligned} \langle \text{ओ}_a|\text{ओ}^a\rangle &= \lim_{x\to\infty}\left(\frac{x}{x^2}\right)^{2\Delta} j_{ca}\left(\frac{x}{x^2}\right)\left\langle 0\middle|\hat{\text{ओ}}^{a\dagger_{\text{NR}}}\left(\frac{x}{x^2}\right)\hat{\text{ओ}}^b(0)\middle|0\right\rangle \\ &= \lim_{x\to\infty}\left(\frac{x}{x^2}\right)^{2\Delta} j_{ca}\left(\frac{x}{x^2}\right)\left(c_{\hat{\text{ओ}}}\,\frac{j^{ab}(x/x^2)}{(x/x^2)^{2\Delta}}\right) = c_{\hat{\text{ओ}}}\,\delta^b_c \end{aligned} \tag{3.18}$$

By defining BPZ conjugation and the dual states, we can now extract many more fundamental properties of CFTs. Taking the BPZ conjugate of the general expression $\hat{Q}_\epsilon$ of Nötherian charges given in (3.10), we have $\hat{Q}_\epsilon^{\dagger_{\text{BPZ}}}$ given by:

$$\hat{Q}_\epsilon^{\dagger_{\text{BPZ}}} = -\hat{Q}_{\hat{\mathcal{J}}(\epsilon^\nu)^*\hat{\mathcal{J}}^{-1}} \tag{3.19}$$

Breaking this down into the BPZ conjugates of the individual elements of the algebra given in (1.10), we note that $\hat{D}$ is BPZ-Hermitian, since the eigenbases of the foliation Hilbert spaces are the eigenstates of $\hat{D}$. Meanwhile, for $\hat{L}_{\mu\nu}$, $\hat{P}_\mu$. and $\hat{K}_\mu$, we note that the expression $\hat{\mathcal{J}}(\epsilon^\nu)^*\hat{\mathcal{J}}^{-1}$ is defined by how it acts on a general function $f(x)$. (Here, $\hat{\mathcal{J}}$ is, as before the sphere inversion operator defined by (1.7).) The composite action of $\hat{\mathcal{J}}$ on functions is defined by $\big(\hat{\mathcal{J}}f\big)(x) = f\big(\hat{\mathcal{J}}(x)\big)$. This gives $\hat{L}_{\mu\nu}^{\dagger_{\text{BPZ}}}$, $\hat{P}_\mu^{\dagger_{\text{BPZ}}}$, and $\hat{K}_\mu^{\dagger_{\text{BPZ}}}$, which we show below. All in all, the BPZ conjugates of the individual elements of the algebra given in (1.10) are:

$$\hat{D}^{\dagger_{\text{BPZ}}} = \hat{D}, \quad \hat{L}_{\mu\nu}^{\dagger_{\text{BPZ}}} = -\hat{L}_{\mu\nu}, \quad \hat{P}_\mu^{\dagger_{\text{BPZ}}} = \hat{\mathcal{J}}\hat{P}_\mu\hat{\mathcal{J}}^{-1} = \hat{\mathcal{J}}\hat{P}_\mu\hat{\mathcal{J}} = \hat{K}^\mu, \quad \hat{K}_\mu^{\dagger_{\text{BPZ}}} = \hat{P}_\mu \tag{3.20}$$

Finally, as in QM, perturbative QFT, and quantum statistical mechanics; we need to define an operator product ordering method to deal with operators defined at two different points in the foliation. In QM, QFT, and quantum stat mech; this was the time-ordering product, which helped us enforce causality by

putting the earlier operators before the later ones. Since we define the foliation in terms of spheres with larger radii corresponding to later times, the corresponding operation is the radial-ordered product:

$$\mathcal{R}\{\hat{\text{अ}}(z_1)\,\hat{\text{ब}}(z_2)\} = \begin{cases} \hat{\text{अ}}(z_1)\,\hat{\text{ब}}(z_2), & |z_1| > |z_2| \\ \hat{\text{ब}}(z_2)\,\hat{\text{अ}}(z_1), & |z_2| > |z_1| \end{cases} \tag{3.21}$$

With the radial ordering defined in this way, we can then define the contour integral of a commutator $\left[\hat{\text{अ}}, \hat{\text{ब}}\right]$ of two operators $\hat{\text{अ}}$ and $\hat{\text{ब}}$:

$$\begin{aligned} \oint \mathrm{d}^d z_1 \left[\hat{\text{अ}}(z_1), \hat{\text{ब}}(z_2)\right] &= \oint_{|z_1|>|z_2|} \mathrm{d}^d z_1\, \hat{\text{अ}}(z_1)\,\hat{\text{ब}}(z_2) - \oint_{|z_1|>|z_2|} \mathrm{d}^d z_1\, \hat{\text{ब}}(z_2)\,\hat{\text{अ}}(z_1) \\ &= \oint_{\mathcal{C}(z_2)} \mathrm{d}^d z_1\, \mathcal{R}\{\hat{\text{अ}}(z_1)\,\hat{\text{ब}}(z_2)\} \end{aligned} \tag{3.22}$$

Here, $\mathcal{C}(z_2)$ is the contour around z_2 that corresponds to the *difference* between the $|z_1| > |z_2|$ and $|z_2| > |z_1|$ contours, as depicted in Figure 6 (modified from [1]).

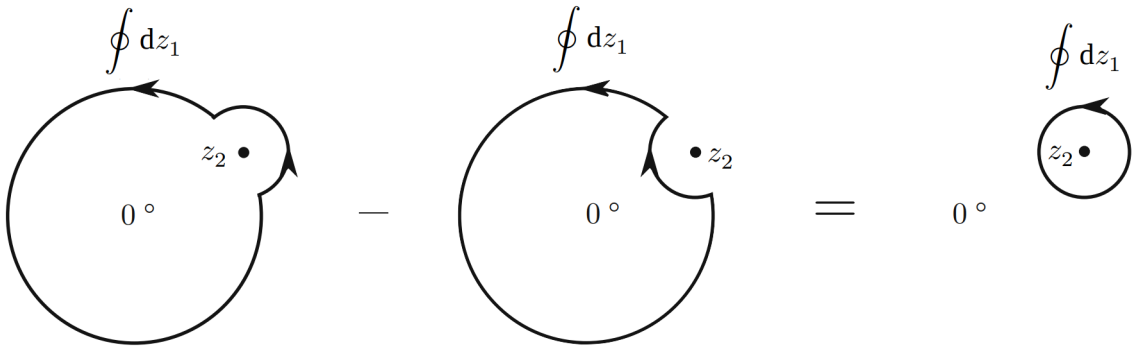

Figure 6: Contour $\mathcal{C}(z_2)$ of integration for the expression of a commutator. Modified from [1].

3.3 Dimension Bounds from Unitarity Properties

Using the BPZ conjugation and the dual states defined in §3.2, we can calculate further important properties of n-point correlators. In particular, we can extract bounds on various properties from the requirement on unitarity, appropriately known as the unitarity bounds.

For a unitary CFT, we require that the norm of states be non-negative; i.e., we have $\langle\psi|\psi\rangle \geq 0$. (Non-unitary CFTs also exist; they're apparently pretty common in condensed matter [2].) A given in-state $|\psi\rangle$ is generated by a bunch of operators acting on the vacuum at $t < 0$ (where this process corresponds to operator insertion in the interior of the sphere, as discussed in §3.1):

$$|\psi\rangle = \hat{\text{ओ}}(-t_i) \cdots \hat{\text{ओ}}(-t_n)\,|0\rangle \tag{3.23}$$

(Here, $t_i > t_n$.) The corresponding out-state is given by:

$$\langle\psi| = \left(\hat{\text{ओ}}(-t_i)\cdots\hat{\text{ओ}}(-t_n)\,|0\rangle\right)^{\dagger_{\text{BPZ}}} = \langle 0|\,\hat{\text{ओ}}^{\dagger_{\text{BPZ}}}(t_n)\cdots\hat{\text{ओ}}^{\dagger_{\text{BPZ}}}(t_i) \tag{3.24}$$

Thus, the condition $\langle\psi|\psi\rangle \geq 0$ is equivalent to requiring that the path integral of a time-inversion-invariant product of operators be non-negative, referred to as reflection positivity. This is depicted in Figure 7, taken from [3, 4].

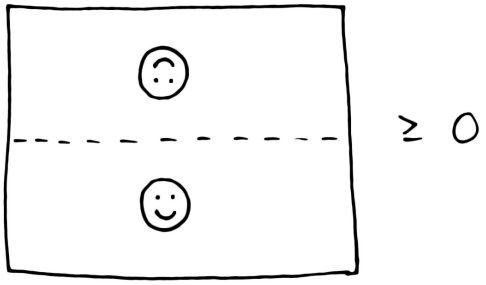

Figure 7: "Reflection positivity" of the path integral. Requiring that the norm of states be non-negative (i.e., that $\langle\psi|\psi\rangle \geq 0$) is equivalent to requiring that the path integral of a time-reversal-invariant product of operators be non-negative. Taken from [3, 4]. This figure doesn't add any understanding, really, but it was too fun not to include.

Using the BPZ conjugations of the conformal algebra generators (3.20), we can extract unitarity bounds straight away. Examining the norm $\left|\hat{P}_\mu\,|\Delta^a\rangle\right|^2$ of the first descendant $\hat{P}_\mu\,|\Delta^a\rangle$ of $|\Delta^a\rangle$, reflection positivity gives us $\left|\hat{P}_0\,|\Delta^a\rangle\right|^2 \geq 0$. Expanding the norm, we have:

$$\left|\hat{P}_0\,|\Delta^a\rangle\right|^2 = \left(\hat{P}_\nu\,|\Delta^b\rangle\right)^{\dagger_{\text{BPZ}}}\hat{P}_\mu\,|\Delta^a\rangle = \langle\Delta_b|\hat{K}^\nu\,\hat{P}_\nu|\Delta^a\rangle = \begin{cases} 2\Delta\delta^\mu_\nu & s\ (\text{spin of }\ |\Delta\rangle) = 0 \\ 2\Delta\delta^\nu_\mu\delta^a_b - 2S^{\nu a}_{\mu b} & s > 0 \end{cases} \tag{3.25}$$

If $|\Delta^a\rangle$ has no spin, then combining the inner product result $2\Delta^\mu_\nu$ with the unitarity bound $\langle\Delta|\Delta\rangle \geq 0$ directly gives the bound $\Delta \geq 0$. (For scalar operators, we drop δ^a_b, since a and b are spin indices.) If we have $|\Delta^a\rangle$ living in the irreducible representation $\mathcal{R}$ of $\mathsf{SO}(d)$ with spin s, meanwhile, then $\hat{P}_\mu\,|\Delta^a\rangle$ lives in the $\square\otimes\mathcal{R}$ representation, where $\square$ is (the Young tableau corresponding to) the vector representation. Imposing the unitarity condition $\langle\psi|\psi\rangle \geq 0$, we have the condition on the conformal dimension Δ given by:

$$\Delta \geq \max\left\{\lambda\left(S^{\nu a}_{\mu b}\right)\right\} \tag{3.26}$$

Here, $\lambda\left(S^{\nu a}_{\mu b}\right)$ represents the set of eigenvalues of $S^{\nu a}_{\mu b}$; i.e., Δ must be greater than or equal to the maximum eigenvalue of $S^{\nu a}_{\mu b}$. We can expand this out in terms of the generator of rotations in $\square$, which we denote $\left(\varpi^{\alpha\beta}\right)_{\mu\nu} := \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$:

$$S^{\nu a}_{\mu b} = \frac{1}{2}\sum_{\alpha,\,\beta}\left(\varpi^{\alpha\beta}\right)^\nu_\mu\left(S_{\alpha\beta}\right)^\nu_\mu = \sum_{\substack{A=\alpha\beta \\ \alpha<\beta}}\left(\varpi^A\right)^\nu_\mu\left(S_A\right)^\nu_\mu \tag{3.27}$$

Here, the index $A = \alpha\beta$ with $\alpha < \beta$ lumps together the transformation in $\square$ and the contraction between ϖ and S; this serves as an adjoint representation index for $\mathsf{SO}(d)$ transformations. If we treat ϖ^A and S_A as operators on $\square \otimes \mathcal{R}$, then the sum becomes:

$$\begin{aligned}\sum_A \varpi^A \otimes S_A &= \frac{(\varpi \otimes \mathbb{1} + \mathbb{1} \otimes S)^2 - (\varpi \otimes \mathbb{1})^2 - (\mathbb{1} \otimes S)^2}{2} \\ &= \frac{-C_2(\square \otimes \mathcal{R}) + C_2(\square) \otimes \mathbb{1} + \mathbb{1} \otimes C_2(\mathcal{R})}{2}\end{aligned} \tag{3.28}$$

In the case that $\mathcal{R}$ is the spin s traceless symmetric representation V_s, we have $C_2(\mathcal{R})$ given by:

$$C_2(V_s) = s(s + d - 2) \tag{3.29}$$

The maximal eigenvalue of $\varpi^A S_A$ (which corresponds to the lowest possible value of the conformal dimension) corresponds to the minimal quadratic Casimir invariant of the $\square \otimes \mathcal{R} = \square \otimes V_s$ representation of $\mathsf{SO}(d)$:

$$\square \otimes V_s = \begin{cases} V_{s-1} \bigoplus_i \mathcal{R}_i \cdots & s > 0 \\ V & s = 0 \end{cases} \tag{3.30}$$

(Here, $\{\mathcal{R}_i\}$ are irreducible representations with larger quadratic Casimir values.) By plugging (3.29) into (3.28), then plugging that result into (3.27), and finally plugging *that* result into (3.26), we get a bound on the dimension Δ:

$$\Delta \geq \frac{-C_2(V_{s-1}) + C_2(\square) + C_2(V_s)}{2}, \therefore \quad \Delta \geq \begin{cases} 0 & s \text{ (spin of } |\Delta\rangle) = 0 \\ s + d - 2 & s > 0 \end{cases} \tag{3.31}$$

(We note that the unitarity bound $s + d - 2$ gives the same lower bound (i.e., $\Delta > 0$); it's just separated because the calculation that was done is exclusive to $s > 0$.) Doing the same calculation again for $\left\langle \Delta \middle| \hat{K}^2 \hat{P}^2 \middle| \Delta \right\rangle = \left\langle \Delta \middle| \hat{K}_\nu \hat{K}^\nu \hat{P}^\mu \hat{P}_\mu \middle| \Delta \right\rangle$ gives:

$$\Delta = 0 \text{ if } |\Delta\rangle = |\mathbb{1}\rangle; \quad \Delta \geq \frac{d-2}{2} \text{ if } |\Delta\rangle \neq \mathbb{1} \tag{3.32}$$

Without further specifying the theory (e.g., by imposing supersymmetry or specifying that this is a 2D CFT), no further unitarity bounds can be extracted by going to higher-level descendants [24]. Thus, the unitarity bounds we get by virtue of conformal invariance generally are:

$$\begin{aligned} &\Delta = 0 && |\Delta\rangle = |\mathbb{1}\rangle; \\ &\Delta \geq \begin{cases} \frac{d-2}{2} & |\Delta\rangle \neq |\mathbb{1}\rangle, \ s = 0; \\ s + d - 2 & |\Delta\rangle \neq |\mathbb{1}\rangle, \ s > 0 \end{cases} \end{aligned} \tag{3.33}$$

For operators *at* this bound, we note that (at least) one of the descendants in the conformal family (2.9)

will be zero. For $|\mathbb{1}\rangle$, all of the descendants are zero. For a scalar, $\Delta = (d-2)/2$, and the null state is $\left(\hat{P}\right)^2 |\Delta\rangle = 0$. As an operator equation, this corresponds to $\hat{\partial}^2 \hat{\mathcal{O}}_\Delta = 0$, i.e., the Klein-Gordon equation; which is a free subsector decoupled from the rest of the CFT. Meanwhile, for a spin-s operator, the null state is $\hat{P}_{\mu_1} |\Delta^{\mu_1\mu_2\cdots\mu_s}\rangle$. This corresponds to the equation $\partial_{\mu_1} \hat{\mathcal{O}}^{\mu_1\mu_2\cdots\mu_s}$, which is simply the equation for a conserved current.

4 Basic Properties of Conformal Field Theories in 2D

Two-dimensional CFTs are especially rich in their structure for a variety of reasons; here, we discuss some of the most basic reasons and their consequences. This section primarily follows [1, 2, 25, 26].

4.1 The Witt Algebra

In 2D, (1.3) reduces to the familiar form of the Cauchy-Riemann equations in $\mathbb{C}^1 \cong \mathbb{R}^2$:

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \tag{4.1}$$

Of course, in $\mathbb{C}^1$, the Cauchy-Riemann equations came about as a result of the conditions we required for differentiability. (4.1) gives a powerful corollary: the conformal group in 2D is the set of all complex analytic functions! This directly tells us that the algebra is infinite dimensional, because we have an infinite number of generators: these are simply the coefficients for the Laurent-Madhava series expansion for ϵ_μ. The power of CFTs in 2D is a consequence of the infinite dimensionality of the conformal algebra, and in fact was already hinted at by the way (1.5a) was derived.

As always for holomorphic / antiholomorphic functions functions, we can use the Wirtinger / holomorphic coordinates $(z, \bar{z})$ to separate the z and $\bar{z}$ pieces and look only at z, with the results being the same for $\bar{z}$. (The z and $\bar{z}$ pieces are respectively called the chiral / left-handed / holomorphic piece and antichiral / right-handed / antiholomorphic piece.) Explicitly, then, considering the infinitesimal conformal transformation $f(z) = z + \epsilon(z)$ (with $\epsilon(z) \ll 1$), we have:

$$z' = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}) \tag{4.2}$$

We can define the generator ℓ_n as corresponding to the nth order of the expansion:

$$\ell_n = -z^{n+1} \partial_z \tag{4.3}$$

The Laurent-Madhava modes $\{\ell_n\}$ generate the conformal algebra in 2D, known as the Witt algebra (which we denote $\mathfrak{Witt}$):

$$\begin{gathered} [\ell_m, \ell_n] = (n+1)\, z^{m+1} z^n \partial_z - (m+1)\, z^{n+1} z^m \partial_z = (n-m)\, z^{m+n+1} \partial_z = (m-n)\, \ell_{m+n} \\ [\ell_m, \bar{\ell}_n] = 0 \end{gathered} \tag{4.4}$$

Examining the commutation relation for $\ell_m = \ell_0$, we have:

$$[\ell_0, \ell_n] = -n\ell_n \tag{4.5}$$

As a result, ℓ_n is a lowering operator for ℓ_0 when $n > 0$, and ℓ_n is a raising operator for ℓ_0 when $n < 0$. Meanwhile, the second expression, $\left[\ell_m, \bar{\ell}_n\right] = 0$, immediately tells us that we have *two* copies of the Witt algebra which are completely disjoint, corresponding to the holomorphic and antiholomorphic pieces. This is just a direct consequence of these pieces generally being disjoint for holomorphic / antiholomorphic functions.

The ℓ_{-1} element is particularly interesting, since this directly corresponds to the z derivative:

$$\ell_{-1} = -z^{-1+1}\,\partial_z = -\partial_z; \quad \therefore\ \ell_{-1}\,\hat{\text{ओ}} = \frac{\partial\hat{\text{ओ}}}{\partial z} \tag{4.6}$$

A direct consequence of this is that each element of the Witt algebra is in fact the (scaled) derivative of the preceding element, which we can see by examining $\partial_z\left[\ell_n\,\hat{\text{ओ}}\right]$ and applying the Witt algebra commutation relation:

$$\begin{gathered}
\frac{\partial}{\partial z}\left[\ell_n\,\hat{\text{ओ}}\right] = \frac{\partial \ell_n}{\partial z}\hat{\text{ओ}} + \ell_n\frac{\partial\hat{\text{ओ}}}{\partial z} \\
\frac{\partial}{\partial z}\left[\ell_n\,\hat{\text{ओ}}\right] = \ell_{-1}\,\ell_n\,\hat{\text{ओ}} = -(n+1)\,\ell_{n-1}\,\hat{\text{ओ}} + \ell_n\,\ell_1\,\hat{\text{ओ}} = -(n+1)\,\ell_{n-1}\,\hat{\text{ओ}} + \ell_n\,\frac{\partial\hat{\text{ओ}}}{\partial z} \\
\implies \frac{\partial \ell_n}{\partial z} = -(n+1)\,\ell_{n-1}
\end{gathered} \tag{4.7}$$

4.2 The Subalgebra $\mathfrak{sl}(2,\mathbb{C})$ and Hermiticity of Witt Generators

Although $\{\ell_n\}$ generates the full Witt algebra, these generators are not defined everywhere. In particular, there's an ambiguity in the definition at the origin. As a result, we must extend the definition of ℓ_n from $\mathbb{R}^2$ to the conformal compactification of $\mathbb{R}^2$, i.e. the Riemann sphere. Unfortunately, even doing this doesn't fully help: ℓ_n is only well-defined at the origin for $n \geq -1$, and if we map $z \mapsto 1/w$, we see that ℓ_n is only defined at the origin for $n \leq 1$.

As a result, while *local* conformal transformations are generated by any analytic function, *global* conformal transformations are only generated by $\{\ell_{-1},\, \ell_0,\, \ell_1\}$; this forms the subalgebra $\mathfrak{sl}(2,\mathbb{C})$ of the Witt algebra. In particular, we have:

- Translations $z \mapsto z + b$ generated $\ell_{-1} = -z^0 = -\partial_z$.
- Dilations $z = r\mathrm{e}^{\mathrm{i}\phi} \mapsto \alpha\mathrm{e}^{\mathrm{i}\phi}$ generated by $\ell_0 + \bar{\ell}_0 = -z\,\partial_z - \bar{z}\,\partial_{\bar{z}} = -r\,\partial_r$.
- Rotations $z = r\mathrm{e}^{\mathrm{i}\phi} \mapsto \mathrm{e}^{\mathrm{i}(\phi+\theta)}$, generated by $\mathrm{i}\left(\ell_0 - \bar{\ell}_0\right) = -\mathrm{i}z\,\partial_z + \mathrm{i}\bar{z}\,\partial_{\bar{z}} = -\partial_\phi$.
- SCTs (corresponding to translations for $z \coloneqq -1/z$ generated by $\ell_1 = -z^2\,\partial_z$.

We immediately note that since dilations and rotations are generated by $\ell_0+\bar{\ell}_0$ and $\mathrm{i}(\ell_0-\bar{\ell}_0)$, respectively, and the generators of both are Hermitian operators, we have ℓ_0 and $\bar{\ell}_0$ both as Hermitian themselves. (Here, we use again the non-radial Hermitian conjugate as our referential starting point, and again use the subscript NR.) Applying their Hermiticity properties to the Witt algebra, using the Hermiticity property of the commutator, also gives an expression for the Hermitian conjugate of the other generators. Combining all of these, we have:

$$\ell_0 = \ell_0^{\dagger_{\mathrm{NR}}}, \quad \bar{\ell}_0 = \bar{\ell}_0^{\dagger_{\mathrm{NR}}}, \quad \ell_n^{\dagger_{\mathrm{NR}}} = \ell_{-n}, \quad \bar{\ell}_n^{\dagger_{\mathrm{NR}}} = \bar{\ell}_{-n} \tag{4.8}$$

Denoting the eigenvalues of ℓ_0 and $\bar{\ell}_0$ as h and $\bar{h}$ respectively, we note that since $\ell_0 + \bar{\ell}_0$ generates dilations, the sum of their eigenvalues must correspond to the conformal dimension. Similarly, since $\mathrm{i}(\ell_0 - \bar{\ell}_0)$ generates rotations, the difference of their eigenvalues must correspond to the spin:

$$\Delta = h + \bar{h}, \quad s = h - \bar{h} \tag{4.9}$$

In §3.1, we used the dilation operator to move between different foliations of our spacetime. As a result, we'd very much like to interpret the dilation operator as the Hamiltonian. In the time-honoured tradition of asserting what we want to be true, for now I'll simply assert that we *can* do this; the validity of this (after radially quantising) will be shown explicitly in §4.6. Finally, we note that we could have directly seen that $\{\ell_1, \ell_0, \ell_{-1}\}$ form a closed subalgebra, by directly applying (4.4):

$$[\ell_{\pm 1}, \ell_0] = \pm \ell_{\pm 1}, \quad [\ell_1, \ell_{-1}] = 2\ell_0 \tag{4.10}$$

The finite transformations generated by the $\mathfrak{sl}(2, \mathbb{C})$ subalgebra are, unsurprisingly, none other than the familiar Möbius transformations; which are given for a, b, c, $d \in \mathbb{C}$ by:

$$z \mapsto \frac{az + c}{bz + d} \tag{4.11}$$

4.3 The Virasoro Algebra

Since the Witt algebra is infinite dimensional, as a Lie algebra it admits a central extension. In general, the central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ of $\mathfrak{g}$ by $\mathbb{C}$ is defined for $\tilde{x}$, $\tilde{y} \in \tilde{\mathfrak{g}}$; x, $y \in \mathfrak{g}$, $c \in \mathbb{C}$, and a bilinear function $p : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by:

$$[\tilde{x}, \tilde{y}]_{\tilde{\mathfrak{g}}} = [x, y]_{\mathfrak{g}} + cp(x, y), \quad [\tilde{x}, c]_{\tilde{\mathfrak{g}}} = 0, \quad [c, c]_{\tilde{\mathfrak{g}}} = 0 \tag{4.12}$$

In general, central extensions are related to projective representations. In particular, they allow us to extend a classical symmetry algebra to include the quadratic Casimir operators of the quantum Lie algebra, thus giving us the full symmetry algebra of a given theory. For the Witt algebra, the central extension is the Virasoro algebra, denoted $\mathfrak{Vir}$. Applying (4.12), we start with the Virasoro algebra given by the expression $[L_m, L_n] = (m-n)L_{m+n} + cp(m, n)$. From the antisymmetry of the Lie bracket and the Jacobi identity, we can get a recursion relation for $p(n, -n)$ (the first equality below), which we can

then get an explicit expression for using the normalisation[7] $p(2,-2) = 1/2$:

$$p(n,-n) = \frac{n+1}{n-2}\, p(n-1,\,-n+1) \xrightarrow[p(2,-2)=1/2]{} \frac{1}{2}\binom{n+1}{3} = \frac{(n+1)(n-1)n}{12} \tag{4.13}$$

This gives us an explicit expression for the Virasoro algebra, in terms of its central charge c:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{(m+n)0}\,\mathbb{1}, \quad \left[L_m, \bar{L}_n\right] = 0, \quad [L_m, \mathbb{1}] = 0 \tag{4.14}$$

The final expression is a bit obvious, but a direct consequence of this is that the elements of the Virasoro algebra all commute with the central charge. As will be discussed later in this section as well as in §4.5, the central charge encodes key properties of the specific CFT model we're looking at. Explicitly, the Witt algebra is the algebra of the conformal symmetry (and holds even for *classical* conformal symmetry), whereas the Virasoro algebra is the specific algebra of the CFT we're looking at.

Since the Virasoro algebra is the central extension of the Witt algebra, it also inherits the properties the Witt algebra received from being comprised of Laurent-Madhava modes of holomorphic functions. In particular, we inherit the non-radial Hermiticity properties (4.8) as well as the decomposition into the disjoint holomorphic and antiholomorphic Virasoro algebras. A direct consequence of the Hermiticity properties is that the central charges are real numbers (i.e., $c, \bar{c} \in \mathbb{R}$); in particular, for *unitary* CFTs, we have $c, \bar{c} \in \mathbb{R}^+$. Additionally, since $p(m,\,n) = 0$ for $m,\,n = -1,\,0,\,1$; the $\{L_{-1},\,L_0,\,L_1\}$ elements of the Virasoro algebra still generate the same global conformal transformations that the corresponding elements of the Witt algebra do[8]. Finally, translating the BPZ conjugation defined in §3.2 into terms amenable to Virasoro generators, we see that (3.14) corresponds to the definition of the inner product given by:

$$\langle \क, \र \rangle := \langle \क(\infty)\, \र(0) \rangle := \lim_{z\to\infty} z^{2\Delta_{\क}} \langle \क(z)\, \र(0) \rangle \tag{4.15}$$

(Here, $\Delta_{\क}$ is the dimension of $\hat{\क}$.)

Given that none of these discussions relied on the central charge, we may reasonably ask what the point of it is. In fact, the central charge is an *essential* link between CFTs and field theories that we might encounter in QFT or stat mech. In particular, the central charge is a central component of unitarity properties of the theory, discussed more in §4.5. In that section, the fact that the central charge helps determine the unitarity properties shows that the central charge represents specific properties of the theory under investigation: different CFTs will all have conformal invariance (and thus, will have all of the properties discussed throughout this paper), but will have different central charges. Indeed, as will be shown in that section, the central charge helps establish whether or not the theory is unitary to begin with; thus, the central charge serves as a link not only to unitary CFTs but nonunitary ones as well.

In addition to this link, the central charge plays a key role in understanding the renormalisation group (RG) properties of the CFT. The Wilsonian / functional integration approach to the RG [27–32] sees

[7] Apparently, this normalisation comes from wanting a specific value for the central charge of the free boson CFT model [1]. I will freely admit I have no familiarity with the free boson CFT model.

[8] This is just a consequence of the fact that finite-dimensional Lie algebras (here, the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra) have (has) no central extensions.

the process of renormalisation as integrating out short-distance (high-momentum) degrees of freedom to obtain the effective theory at a given length scale, with the renormalisation group equations (namely, the Callan-Symanzik equation) then telling us how the various couplings in the theory change as we change the length scale. By integrating out the higher momentum scales, we inherently lose information about the system. This general conception is made concrete by the c-theorem [33]. The original statement of this theorem says that every 2D QFT has a positive real function $C(\text{ચ}, M)$, dependent on the coupling constants ચ and the renormalisation scale M, which decreases monotonically as the RG flows to the fixed point. *At* the fixed point, $C(\text{ચ}, M)$ is a constant, and is exactly the central charge of the corresponding CFT. The C-theorem was proved for 2D QFTs; an analogous version (the a-theorem [34, 35]) has been largely shown to be valid, although certain aspects of it remain an open question of investigation.

4.4 Primaries in 2D CFTs and the Verma Module

In 2D, the definition (2.7) for primary fields corresponds to the requirement that $\hat{\text{ધ}}$ transform under any *infinitesimal* conformal transformation $z \mapsto f(z)$ as:

$$\hat{\text{ધ}}(z,\bar{z}) \mapsto \hat{\text{ધ}}'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \hat{\text{ધ}}\big(f(z),\, \bar{f}(\bar{z})\big) \tag{4.16}$$

We can immediately recognise ∂_z and $\partial_{\bar{z}}$ as L_0 and $\bar{L}_0$; thus, this expression of the original definition (2.7) automatically translates into a statement in terms of the eigenvalues h and $\bar{h}$ of L_0 and $\bar{L}_0$ respectively. As discussed in §4.2, the conformal dimension Δ is given by $\Delta = h + \bar{h}$; we also refer to the conformal dimension as $(h, \bar{h})$. If $\hat{\text{ધ}}$ only satisfies the transformation property (4.16) for *global* conformal transformations, then $\hat{\text{ધ}}$ is a quasi-primary field instead.

Examining the action of the infinitesimal conformal transformation $f(z) = z + \epsilon(z)$ with $\epsilon(z) \ll 1$ and taking the Taylor-Madhava expansions of $\epsilon(z)$ and (4.16), we have the action of an infinitesimal conformal transformation on a primary operator given by:

$$\hat{\text{ધ}} \mapsto \left(h\frac{\partial \epsilon}{\partial z} + \epsilon\frac{\partial}{\partial z} + \bar{h}\frac{\partial \bar{\epsilon}}{\partial \bar{z}} + \bar{\epsilon}\frac{\partial}{\partial \bar{z}}\right)\hat{\text{ધ}} \tag{4.17}$$

Additionally, as mentioned in §4.1, L_n are lowering operators for L_0 when $n > 0$ and raising operators for L_0 when $n < 0$ (with the same holding true for $\bar{L}_n$ and $\bar{L}_0$). Since we have $L_0 + \bar{L}_0$ as the dilation operator, whose eigenvalues are the conformal dimension, we can immediately determine the effect of L_0 and L_n on $|\text{ધ}\rangle$ for $n > 0$:

$$L_0 |\text{ધ}\rangle = \Delta_{\text{ધ}} |\text{ધ}\rangle\,, \quad L_n |\text{ધ}\rangle = 0 \quad (n > 0) \tag{4.18}$$

Examining $\|L_{-m}|\text{ધ}\rangle\|$ for a generic primary $\hat{\text{ધ}}$ with conformal dimension $\Delta_{\text{ધ}}$, the fact that the norm must be positive definite gives $\|L_{-m}|\text{ધ}\rangle\| \geq 0$. Expanding the norm out, we have:

$$\|L_{-m}|\text{ધ}\rangle\|^2 = \big\langle \text{ધ}\big| {L_{-m}}^{\dagger_{\text{NR}}} L_{-m}\big|\text{ધ}\big\rangle = \big\langle \text{ધ}\big| {L_{-m}}^{\dagger_{\text{NR}}} L_{-m}\big|\text{ધ}\big\rangle = \frac{c}{12}(m^3 - m)\,\langle \text{ધ}|\text{ધ}\rangle = \frac{c}{12}(m^3 - m) \tag{4.19}$$

(Here, we use the normalisation $\langle \text{ધ}|\text{ધ}\rangle = 1$.) Combining this with $\|L_{-m}|\text{ધ}\rangle\|^2 \geq 0$, we see that $c \in \mathbb{R}^+$.

Since we now have $L_n |\mathfrak{q}\rangle = 0$ for $n > 0$, and we have $L_n |\mathfrak{q}\rangle$ producing a positive semidefinite norm state for $n < 0$, we can immediately determine that the set of all $\left\{(L_n)^{\ell}\right\}$ generates the generalised Verma module identified in (3.5) when applied to $|\mathfrak{q}\rangle$. The conformal multiplet is generated by applying $\left\{(L_n)^{\ell}\right\}$ to the highest-weight state $|\mathfrak{q}\rangle$; thus, the family corresponds to the highest-weight representation of the Virasoro algebra. Here in the 2D case, the representation generated this way is the Verma module, which we can define more explicitly than (3.5) using the Virasoro operators. Explicitly, we define the Verma module $\mathcal{V}_{(h,c)}$ corresponding to the primary $\mathfrak{q}$ with conformal dimension $(h, \bar{h})$ as the highest-weight representation of $\mathfrak{Vir}$ with central charge c, whose basis is given by:

$$\mathcal{V}_{(h,c)} := \text{span}\left\{ \prod_{i=1}^{\ell} L_{-n_i} |\mathfrak{q}\rangle \right\}_{0 \le n_1 \le \cdots \le m_\ell} \tag{4.20}$$

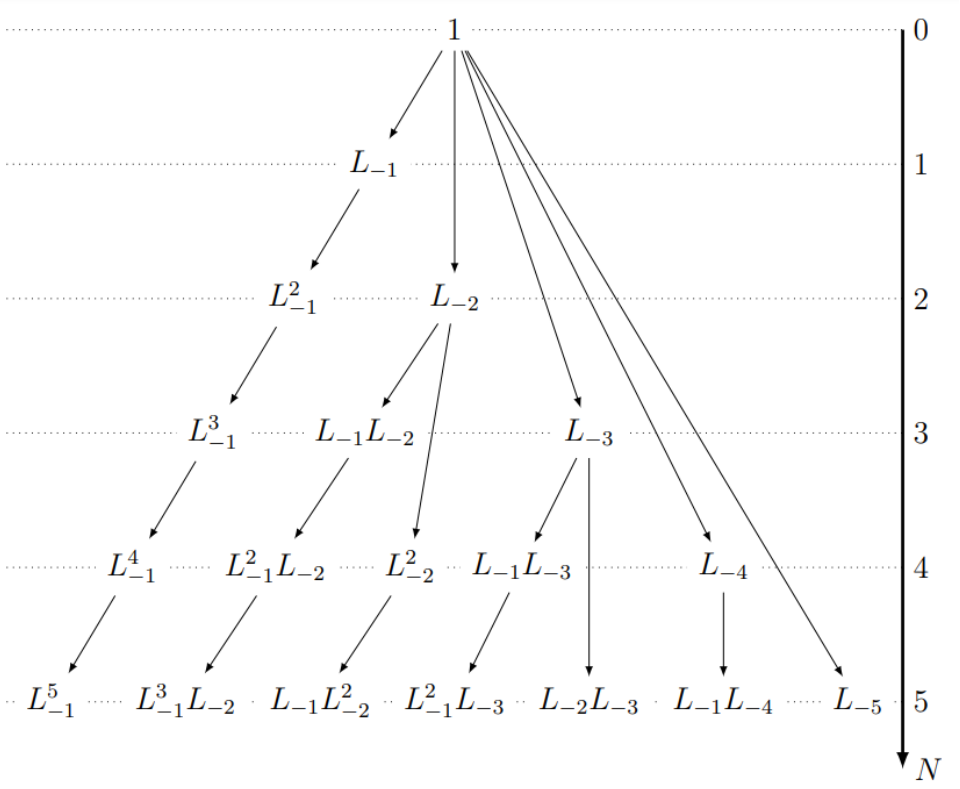

Figure 8: Verma module up to level $N = 5$. Taken from [26].

The definition of the level of the state in (2.9) extends straightforwardly: for the state $\prod_{i=1}^{\ell} L_{-n_i} |\mathfrak{q}\rangle$, the level is given by $N = \sum_{i=1}^{\ell} n_i$, and level $N > 1$ states are descendant states. To make these expressions concrete, the basis of the Verma module up to level 5 is given from [26] in Figure 8.

At first glance, we may think that this chart is missing some entries; for instance, the level 3 states seem to be missing $L_{-2}L_{-1}$. However, from the Virasoro commutation relation (4.14), this state is equivalent to $L_{-2}L_{-1} = L_{-1}L_{-2} - L_{-3}$, and thus this state is a linear combination of other states in the basis. Thus, we can restrict the ordering of Virasoro generators, as in (4.20), to be "smallest n first", as in Figure 8.

We note that this choice is completely arbitrary; we can just as easily order them to be "largest n first", or really any other set that gives a basis following the Virasoro commutation relation.

4.5 Null States and the Kač Determinant

As discussed in §3.3, states that are *at* the unitarity bounds for dimension given in (3.33) have at least one vanishing descendant. The presence of the scalar null state $\hat{P}^2 |\mathfrak{q}\rangle$ indicated a free Klein-Gordon subsector that decoupled from the rest of the CFT. More generally, null states indicate the presence of decoupled subsectors. These correspond to subrepresentations of the Virasoro algebra within the Verma module, meaning that the original Verma module is a reducible representation. As an example, at level 2, a general normalised linear combination of the states is given by:

$$\left(L_{-2} + \zeta {L_{-1}}^2\right) |\mathfrak{q}\rangle = 0 \tag{4.21}$$

If the value of ζ corresponds to a state that exists in the module, then we have a null state and the Verma module $\mathcal{V}_\mathfrak{q}$ is a reducible representation of the Virasoro algebra. To find the null states, we define the matrix $M_N(\Delta_\mathfrak{q}, c)$ at arbitrary level N (which I'm calling the Kač matrix, although I don't know if this is standard usage) by:

$$M_N(\Delta_\mathfrak{q}, c) := \left\langle \mathfrak{q} \middle| \prod_i L_{k_i} \prod_j L_{m_j} \middle| \mathfrak{q} \right\rangle, \quad k_i,\, m_j \geq 0 \tag{4.22}$$

The determinant of this is given for $\alpha_N \in \mathbb{R}^+$, $m \in \mathbb{C}$, and for the number of partitions $p(N - pq)$ of $N - pq$ by:

$$\begin{aligned} \det M_N(\Delta_\mathfrak{q}, c) &:= \alpha_N \prod_{\substack{p,\,q \leq N \\ p,\,q > 0}} \left(\Delta_\mathfrak{q} - \Delta_{\mathfrak{q},\,p,\,q}(c)\right)^{p(N-pq)} \\ \Delta_{\mathfrak{q},\,p,\,q}(m) &:= \frac{\left((m+1)\,p - mq\right)^2 - 1}{4m(m+1)}, \quad m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \end{aligned} \tag{4.23}$$

The Kač determinant tells us the values of $\Delta_\mathfrak{q}$ and c that give rise to decoupled subsectors, but also tells us values of $\Delta_\mathfrak{q}$ and c that are forbidden by unitarity. Specifically, when $\det M_N < 0$, we have negative norm states, which is forbidden for unitary CFTs. This gives us a series of restrictions on the values of c and $\Delta_\mathfrak{q}$:

- For $c > 1$ and $\Delta_\mathfrak{q} \geq 0$, $\det M_N$ is always positive, and thus every such CFT is unitary.
- For $c = 1$, $\det M_N = 0$ for $\Delta_\mathfrak{q} = n^2/4$ (with $n \in \mathbb{Z}$).
- For $c < 1$ and $\Delta_\mathfrak{q} \geq 0$, $\det M_N$ is *only* nonunitary for values of c given for $m \in \mathbb{N}^{\geq 3}$ and the corresponding $\binom{m}{2}$ values of $\Delta_\mathfrak{q}$, p, q by:

$$c = 1 - \frac{6}{m(m+1)}, \quad \Delta_{\mathfrak{q},\,p,\,q}(m) = \frac{\left((m+1)\,p - mq\right)^2 - 1}{4m(m+1)}, \quad 1 \leq p \leq m-1, \quad 1 \leq q \leq m \tag{4.24}$$

  These correspond to intersections of specific curves in the $(c, \Delta_\mathfrak{q})$ plane, depicted in Figure 9 (taken from [1]).

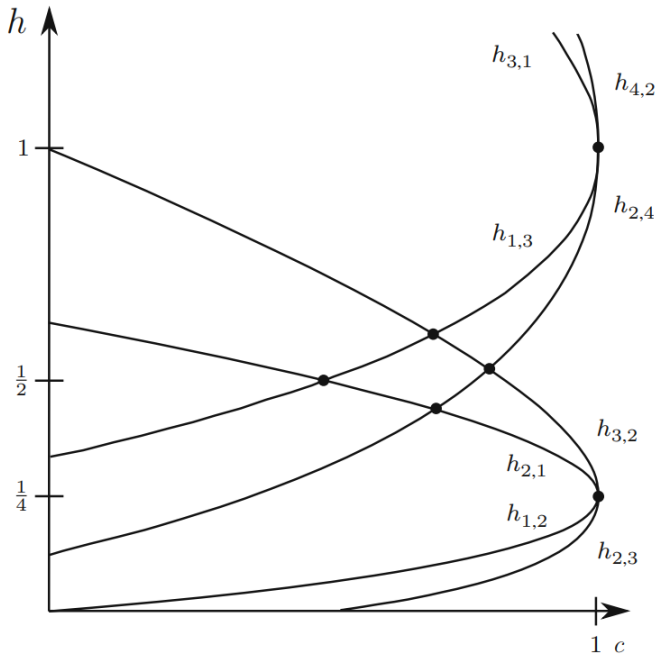

Figure 9: Unitary CFTs for $c < 1$ correspond to the intersections of curves defined by (4.24). The first few such curves are depicted. Here, $\Delta_{\mathfrak{q}}$ is labelled as h and $\Delta_{\mathfrak{q},\,p,\,q}$ is labelled as $h_{p,\,q}$. Taken from [1].

Before continuing, an interesting additional point is worth mentioning about null states. As mentioned in §3.3 and in the beginning of this section, null states correspond to operators whose conformal dimensions are *at* the unitarity bounds given in (3.33). These indicate that the Verma module is a reducible representation of the Virasoro algebra, and thus these states specifically correspond to decoupled subsectors of the Verma module, with at least one vanishing descendant. These subsectors are minimal models, where the spectrum is built out of finitely many irreps of the Virasoro algebra. A valuable property of minimal models is that, since the Virasoro algebra representation is reducible, we can dramatically simplify linear algebraic manipulations. In particular, the resolutions of the identity (3.7) and (3.8) can be simplified into a sum over a *single* conformal family[9]. For a primary operator $\hat{\mathfrak{q}}$ in a minimal model with conformal family $[\hat{\mathfrak{q}}]$, those resolutions of the identity simplify to:

$$\mathbb{1} = \sum_{[\hat{\mathfrak{q}}]} |[\hat{\mathfrak{q}}]\rangle\langle[\hat{\mathfrak{q}}]| = \sum_{i=0}^{\infty} |L_{-i}\hat{\mathfrak{q}}\rangle\langle L_{-i}\hat{\mathfrak{q}}| \tag{4.25}$$

The central charges and conformal dimensions for unitary minimal models are given from the Kač determinant by:

$$c = 1 - \frac{6(p-q)^2}{pq}, \quad \Delta_{\mathfrak{q},\,r,\,s}(p,q) = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \tag{4.26}$$

Here, we have $p, q \in \mathbb{N}^{+2}$ with p and q relatively coprime, $r \in [1, q-1] \subset \mathbb{N}$, and $s \in [1, p-1] \subset \mathbb{N}$. If $|p-q| \neq 1$, we're *guaranteed* to have a nonunitary CFT (i.e., we're guaranteed to have highest-weight

[9] I'm especially grateful to Adam for this clarification in particular.

states of negative norm), with a unitary subrepresentation given by $p = m+2$ and $q = m+3$ with $m \geq 1$. The unitary subrepresentations are unitary minimal models, which have additional properties discussed in §7.2–§7.3. If the *entire* theory is comprised of unitary minimal models, c and $\Delta_{\mathfrak{q}}$ reduce to (4.24). Since c in that situation is a rational positive number less than 1, those theories are called rational CFTs.

As an example of calculating the restrictions on c and $\Delta_{\mathfrak{q}}$ that arise from the presence of null states, we calculate these values for the level 2 null state described in (4.21). In addition to being a simple calculation illustrating the Kač determinant technique, the results will be used in §7.2. Starting with this expression, and applying L_1 to both sides, we have:

$$\begin{gathered}\left([L_1, L_{-2}] + \mathfrak{z}\left[L_1, {L_{-1}}^2\right]\right)|\mathfrak{q}\rangle = (3L_{-1} + \mathfrak{z}(2L_0L_{-1} + 2L_{-1}L_0))\,|\mathfrak{q}\rangle = 0 \\ \left([L_1, L_{-2}] + \mathfrak{z}\left[L_1, {L_{-1}}^2\right]\right)|\mathfrak{q}\rangle = (3 + 2\mathfrak{z}(2\Delta_{\mathfrak{q}} + 1))L_{-1}\,|\mathfrak{q}\rangle = 0\end{gathered} \tag{4.27}$$

Since generally we have $L_{-1}\,|\mathfrak{q}\rangle = 0$ only for $\Delta_{\mathfrak{q}} = 0$, we have $\mathfrak{z} = -3/2(2\Delta_{\mathfrak{q}} + 1)$, giving the level 2 null state as:

$$\left(L_{-2} - \frac{3}{2(2\Delta_{\mathfrak{q}} + 1)}{L_{-1}}^2\right)|\mathfrak{q}\rangle = 0 \tag{4.28}$$

Then, applying L_2 to both sides, we can determine the central charge of the theory where this null state emerges:

$$\begin{gathered}\left([L_2, L_{-2}] + \frac{3}{2(2\Delta_{\mathfrak{q}} + 1)}\left[L_2, {L_{-1}}^2\right]\right)|\mathfrak{q}\rangle = 0 \\ \left(4L_0 + \frac{c}{2}\right)|\mathfrak{q}\rangle - \frac{3}{2(2\Delta_{\mathfrak{q}} + 1)}(L_{-1}[L_2, L_{-1}] + [L_2, L_{-1}]L_{-1}) = 0 \\ \left(4L_0 + \frac{c}{2} - 6 \cdot \frac{3}{2(2\Delta_{\mathfrak{q}} + 1)}\right)|\mathfrak{q}\rangle = \left(4L_0 + \frac{c}{2} - \frac{9}{(2\Delta_{\mathfrak{q}} + 1)}\right)|\mathfrak{q}\rangle = 0\end{gathered} \tag{4.29}$$

This gives the central charge as $c = (2\Delta_{\mathfrak{q}}(5 - 8\Delta_{\mathfrak{q}}))/(2\Delta_{\mathfrak{q}} + 1)$. Thus, we have the level 2 null state occurring *specifically* at this central charge, with the null state given by (4.28).

4.6 Radial Quantisation in 2D

As in §3, to get further properties, we examine the radial quantisation of the theory. From Euclidean space (x^0, x^1), we compactify the space dimension on a circle of radius $R = 1$. Thus, from functions of z, we transform to functions of w, with the transformation defined by:

$$z = \mathrm{e}^w = \mathrm{e}^{x^0} + \mathrm{e}^{\mathrm{i}x^1}, \quad w := x^0 + \mathrm{i}x^1, \quad w = w + 2\pi\mathrm{i} \tag{4.30}$$

For Lorentzian space, we Wick rotate $x^0 \mapsto \mathrm{i}x^0$ as always. This mapping is depicted in Figure 10, taken from [1]. Using this mapping, non-radial Hermitian conjugation maps $z \mapsto 1/\bar{z}$. This then defines our 2D BPZ Hermitian conjugation:

$$\hat{\mathfrak{o}}^{\dagger_{\mathrm{BPZ,\,2D}}}(z, \bar{z}) = \bar{z}^{-2h}\, z^{-2\bar{h}}\, \hat{\mathfrak{o}}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) \tag{4.31}$$

Meanwhile, the Laurent expansion of a field $\mathcal{O}(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ around $z=0$ and $\bar{z}=0$ is given by:

$$\hat{\mathcal{O}}(z,\bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\,\bar{z}^{-\bar{m}-\bar{h}}\,\hat{\mathcal{O}}_{n,\bar{m}} \tag{4.32}$$

Applying the BPZ conjugate and Laurent expanding then gives us:

$$\hat{\mathcal{O}}^{\dagger_{\text{BPZ, 2D}}}(z,\bar{z}) = \bar{z}^{-2h}\,z^{-2h}\sum_{n,\bar{m}\in\mathbb{Z}} z^{n+h}\,\bar{z}^{\bar{m}+\bar{h}}\,\hat{\mathcal{O}}_{n,\bar{m}} = \sum_{n,\bar{m}\in\mathbb{Z}} z^{n-h}\,\bar{z}^{\bar{m}-\bar{h}}\,\hat{\mathcal{O}}_{n,\bar{m}} \tag{4.33}$$

Comparing (4.32) and (4.33) gives us $\left(\hat{\mathcal{O}}_{n,\bar{m}}\right)^{\dagger_{\text{BPZ, 2D}}} = \hat{\mathcal{O}}_{-n,-\bar{m}}$, corresponding to our expectations given the action of Hermitian conjugation on the Witt algebra generators (4.8).

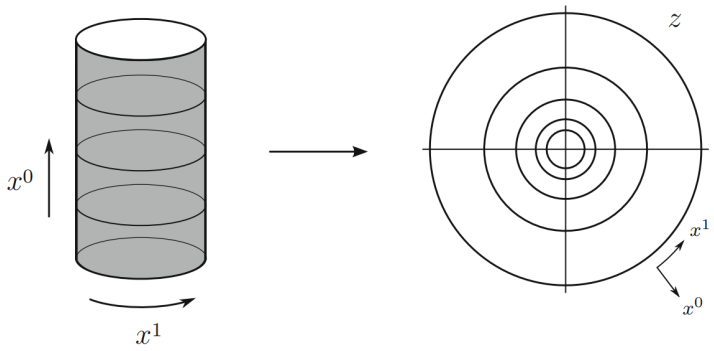

Figure 10: Mapping of the cylinder to the complex plane given by $z = \mathrm{e}^{x^0} + \mathrm{e}^{\mathrm{i}x^1}$. Taken from [1].

In §4.2, we asserted that the dilation operator corresponded to the Hamiltonian, not just as an analogy but rather specifically in the sense that the Hamiltonian is mapped to the dilation operator. We immediately see this as a consequence of the coordinate transformation: time translations $x^0 \mapsto x^0 + a$ correspond to dilations $z \mapsto \mathrm{e}^a z$. In the same way as time translations, we have space translations $x^1 \mapsto x^1 + b$ corresponding to rotations $z \mapsto \mathrm{e}^{\mathrm{i}b} z$. This verifies the equivalency made in §4.2 of the Hamiltoninan and momentum operator explicitly corresponding to the conformal dilation and rotation generators, and allows us to write explicitly:

$$\hat{H} = L_0 + \bar{L}_0, \quad \hat{P}_\mu = \mathrm{i}(L_0 - \bar{L}_0) \tag{4.34}$$

In addition, we note that this transformation maps the infinite past $x^0 = -\infty$ to the origin: $z = \mathrm{e}^{-\infty} = 0$. As discussed in §3.1, we generally foliate spacetime radially with the reference point at the origin. We can combine this with the raising and lowering properties of the generators, which gives us $\hat{\mathcal{O}}_{n,\bar{m}}\,|0\rangle = 0$ for $n > -h$ and $\bar{m} > -\bar{h}$, to get the asymptotic in-state as the reference point for our foliation:

$$|\mathcal{O}\rangle = \lim_{z,\bar{z}\to 0} \hat{\mathcal{O}}(z,\bar{z})\,|0\rangle = \hat{\mathcal{O}}_{-h,-\bar{h}}\,|0\rangle \tag{4.35}$$

Similarly, we can define the asymptotic out-state by taking the BPZ conjugate of the asymptotic in-state, again using the raising and lowering properties of the generators to get $\langle 0|\,\hat{\mathcal{O}}_{n,\bar{m}} = 0$ for $n > h$ and

$\bar{m} > \bar{h}$:

$$\langle \hat{\text{ओ}}| = \lim_{z,\bar{z}\to 0} \langle 0|\, \hat{\text{ओ}}^{\dagger_{\text{BPZ, 2D}}}(z,\bar{z}) = \lim_{\bar{\zeta},\zeta\to\infty} \langle 0|\, \hat{\text{ओ}} = \langle 0|\, \hat{\text{ओ}}_{h,\bar{h}} \tag{4.36}$$

5 The Stress-Energy Tensor and Ward Identities

5.1 The Stress-Energy Tensor

The original statement of conformal invariance, (1.1), told us that the metric was invariant up to a constant positive scale factor $\Lambda \in \mathbb{R}^+$. This gave rise to four symmetries, discussed along with their corresponding currents in §1. In addition to these, conformal invariance as a whole implies the conservation of the stress-energy tensor $T^{\mu\nu}$. This has far-reaching consequences; we briefly discuss some of them here.

Nöther's theorem in general gives us the variation of the action under an infinitesimal transformation, and thus the conserved quantities. Following [5], we start with the action S in curved spacetime, depending on the fields $\{\mathfrak{Ş}_i\}$ and the vielbein e^a_μ as:

$$S\big[\mathfrak{Ş}_i, e^a_\mu\big] = \int \mathrm{d}^d x\, \det e^a_\nu\, \mathcal{L}\big(\mathfrak{Ş}_i, D_\mu \mathfrak{Ş}_i, e^a_\mu\big) \tag{5.1}$$

Here, the vielbein / vierbein / tetrad is a set of n independent vector fields $\big\{e^a_\mu\big\}_{a=1,\cdots,n}$ defined in terms of the basis 1-forms $\mathrm{d}x^\mu$ of the manifold the system lives on. This provides a basis for the manifold which can be easier to deal with. We can define the vielbein and its dual (the dual vielbein / co-vielbein) in terms of the derivatives and 1-forms, as well as the transformations of the vielbeins and fields $\{\mathfrak{Ş}_i\}$ under reparametrisations $x \mapsto y$:

$$e_a = e^\mu_a\, \partial_\mu, \quad e^a = e^a_\mu\, \mathrm{d}x^\mu\,, \quad e^a\, e_b = \delta^a_b; \quad e^a_\mu \mapsto \frac{\partial x^\mu}{\partial y^\nu} e^a_\nu, \quad \mathfrak{Ş}(x) \mapsto \acute{\mathfrak{Ş}}(y) = \mathfrak{Ş}(x) \tag{5.2}$$

We have the variation of the action under an infinitesimal transformation (in coordinate language) given by:

$$\delta S = -\int \mathrm{d}^d x\, j^\mu\, \partial_\mu \epsilon \tag{5.3}$$

We then consider the infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$ given in §1, and consider the variation induced in both the action and the metric due to this transformation. In terms of the coordinate transformation $x \mapsto x + \epsilon$ or, equivalently, the vielbein variation $e^a_\mu \mapsto \delta e^a_\mu$, we have:

$$\delta S = -\int \mathrm{d}^d x \left(T^{\mu\nu} + 2\frac{\delta S}{\delta g^{\mu\nu}}\right)\big(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\big); \quad \delta S = \frac{1}{2}\int \mathrm{d}^d x\, \det e^a_\nu\, T^\mu_a\, \delta e^a_\mu \tag{5.4}$$

For both of these expressions, we define the stress-energy tensor as:

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{5.5}$$

Applying the infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$, we have the corresponding conserved current

for this transformation given by $j_\mu = T_{\mu\nu}\,\epsilon^\nu$. Then, from the conservation of current, $\partial^\mu j_\mu = 0$, we have:

$$\begin{aligned}\partial^\mu j_\mu &= \partial^\mu \left[T_{\mu\nu}\,\epsilon^\nu\right] = \left(\partial^\mu\, T_{\mu\nu}\right)\epsilon^\nu + T_{\mu\nu}\left(\partial^\mu \epsilon^\nu\right) \\ &= 0 + \frac{T_{\mu\nu}\left(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu\right)}{2} = \frac{T_{\mu\nu}\,g^{\mu\nu}(\partial\cdot\epsilon)}{d} = \frac{T^\mu_\mu\,(\partial\cdot\epsilon)}{d} = 0\end{aligned} \tag{5.6}$$

(In the second line, we used (1.1).) As a result, we see that in CFTs, the stress-energy tensor is traceless. Translating this into 2D, this property combines with the conservation of current to get that the only nonvanishing components of the stress-energy tensor are the purely holomorphic piece and the purely antiholomorphic piece:

$$T(z) \coloneqq T_{zz}(z), \quad \bar{T}(\bar{z}) \coloneqq T_{\bar{z}\bar{z}}(\bar{z}) \tag{5.7}$$

(This is, as always, derived in the unabridged version.) Since the set of conformal transformations in 2D is itself the set of all holomorphic functions, which have this decomposition property, it's unsurprising that the stress-energy tensor has the same property. Writing the holomorphic and antiholomorphic sectors in this fashion, however, gives us a further useful expression: we can express $T(z)$ as the Laurent-Madhava series:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{(z-\text{ॐ})^{n+2}} \tag{5.8}$$

(As always, $\bar{T}(\bar{z})$ has the same expression, with the replacements $L_n \to \bar{L}_n$, $z \to \bar{z}$, and $\text{ॐ} \to \bar{\text{ॐ}}$.) Importantly, the stress-energy tensor has a mode for *every* Virasoro generator. As a result, we can invert this expression to get an expression for the Virasoro generators in terms of the holomorphic stress-energy tensor:

$$L_n = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\, z^{n+1}\, T(z) \tag{5.9}$$

Notably, this expansion of $T(z)$ in terms of Virasoro generators, and the expansion of the $\{L_n\}$ in terms of $T(z)$ allows us to automatically read off the product of $T(z)$ and a generic field $\hat{\text{ओ}}(\text{ॐ})$, and allows us to give $L_n\,\hat{\text{ओ}}(\text{ॐ})$ as a contour integral of $T(z)\,\hat{\text{ओ}}(\text{ॐ})$:

$$\mathcal{R}\left\{T(z)\,\hat{\text{ओ}}(\text{ॐ})\right\} = \sum_{n\in\mathbb{Z}} \frac{L_n\,\hat{\text{ओ}}(\text{ॐ})}{(z-\text{ॐ})^{n+2}}, \quad L_n\,\hat{\text{ओ}}(\text{ॐ}) = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z\,(z-\text{ॐ})^{n+1}\, T(z)\,\hat{\text{ओ}}(\text{ॐ}) \tag{5.10}$$

If $\hat{\text{ओ}}$ is a primary field $\hat{\text{पे}}$, this gives the product $\mathcal{R}\{T(z)\,\hat{\text{पे}}(\text{ॐ})\}$ as:

$$\mathcal{R}\left\{T(z)\,\hat{\text{पे}}(\text{ॐ})\right\} = \frac{\Delta_{\text{पे}}\,\hat{\text{पे}}}{(z-\text{ॐ})^2} + \frac{1}{z-\text{ॐ}}\frac{\partial\hat{\text{पे}}}{\partial\text{ॐ}} + f_{\text{reg}}(z) \tag{5.11}$$

Here, f_{reg} is the regular (non-singular) part of this expansion. (5.10) and (5.11) are two examples of operator product expansions (OPEs). The formal structure of OPEs is given in §6.1; due to (5.9), we are able to extract OPEs involving the stress-energy tensor without it.

Finally, we note that under conformal transformations $z \mapsto f(z)$, the stress-energy tensor transforms as:

$$T(z) \mapsto \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S[f(z), z] = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} \frac{1}{\left(\frac{\partial f}{\partial z}\right)^2} \left(\frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z^3} - \frac{3}{2}\left(\frac{\partial^2 f}{\partial z^2}\right)^2\right) \tag{5.12}$$

Here, $S[f(z), z]$ is the Schwartzian derivative, defined by:

$$S[f(z), z] = \frac{1}{\left(\frac{\partial f}{\partial z}\right)^2} \left(\frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z^3} - \frac{3}{2}\left(\frac{\partial^2 f}{\partial z^2}\right)^2\right) \tag{5.13}$$

5.2 Conserved Charges

As in perturbative QFT, the classical conserved current expression $\partial_\mu j^\mu = 0$ is expressed quantum mechanically via a Ward identity. Since the conserved currents can be lumped together into the stress-energy tensor, we can express the Ward identities in terms of the stress-energy tensor as well. Here, I've attempted to combine the approaches in [1–5]. I relied mostly on the latter at first, but the end is entirely from [5] specifically.

In general, we define a conserved charge in d dimensions as the integral of a conserved current j^μ over a Cauchy surface Σ:

$$Q[\Sigma] = \int \mathrm{d}S_\mu \, j^\mu \tag{5.14}$$

Here, a Cauchy surface is a surface intersected by any causal curve at most once; heuristically, it corresponds to the notion of an "equal-time surface" [3, 4, 36] for our purposes. We note that from the translational Ward identity (5.24), $Q[\Sigma]$ defined in (5.14) is invariant under diffeomorphisms of Σ, as long as it does not cross any operator insertions. This is depicted in Figure 11, modified from [3, 4].

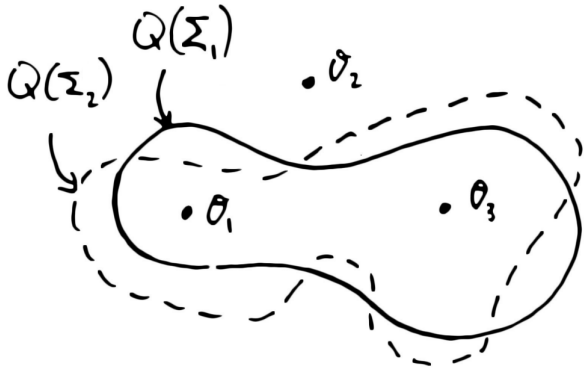

Figure 11: Conserved charges $Q[\Sigma]$ are invariant under diffeomorphisms of the Cauchy surface Σ, as long as they don't cross any operator insertions. Modified from [3, 4].

Applying the coordinate transformation given in (4.30), using $j^\mu = T^{\mu\nu}\epsilon_\nu$ (as per the coordinate expression in (5.4)), and using the fact that in 2D, the stress-energy tensor decomposes into the purely holomorphic and purely antiholomorphic sectors; the expression (5.14) corresponds for 2D CFTs to the

expression:

$$Q = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \left(\mathrm{d}z \, T(z) \, \epsilon(z) + \mathrm{d}\bar{z} \, \bar{T}(\bar{z}) \, \bar{\epsilon}(\bar{z}) \right) \tag{5.15}$$

As always, the conserved charge Q generates an infinitesimal transformation $\delta\hat{\text{ओ}}$ of an operator $\hat{\text{ओ}}$ by examining the commutator. In 2D, this appears as:

$$\begin{gathered} \delta\hat{\text{ओ}} = \left[Q, \hat{\text{ओ}} \right] \\ \left(\delta\hat{\text{ओ}} \right)_{\text{2D}} = \frac{1}{2\pi \mathrm{i}} \left(\oint_{\mathcal{C}} \mathrm{d}z \left[T(z) \, \epsilon(z), \hat{\text{ओ}}(z, \bar{z}) \right] + \oint_{\mathcal{C}} \mathrm{d}\bar{z} \left[\bar{T}(\bar{z}) \, \bar{\epsilon}(\bar{z}), \hat{\text{ओ}}(z, \bar{z}) \right] \right) \end{gathered} \tag{5.16}$$

Applying (3.22), this gives $\left(\delta\hat{\text{ओ}} \right)_{\text{2D}}$ as:

$$\left(\delta\hat{\text{ओ}} \right)_{\text{2D}} = \frac{1}{2\pi \mathrm{i}} \left(\oint_{\mathcal{C}} \mathrm{d}z \, \epsilon(z) \, \mathcal{R}\left\{ T(z) \, \hat{\text{ओ}}(\text{ॐ}, \bar{\text{ॐ}}) \right\} + \oint_{\mathcal{C}} \mathrm{d}\bar{z} \, \bar{\epsilon}(\bar{z}) \, \mathcal{R}\left\{ \bar{T}(\bar{z}) \, \hat{\text{ओ}}(\text{ॐ}, \bar{\text{ॐ}}) \right\} \right) \tag{5.17}$$

For a primary operator $\hat{\text{पि}}$, we already had $\mathcal{R}\left\{ T(z) \, \hat{\text{पि}}(\text{ॐ}) \right\}$ given by (5.11); we see that this can be extracted by applying (4.17) to (5.17).

5.3 Conformal Ward Identities for Translation and Rotation Invariance

In terms of the action (5.1), we have the partition function for the n-point function of generic operators $\left\{ \hat{\text{ओ}}_j \right\}$ given by:

$$Z \left\langle \hat{\text{ओ}}_1(x_1) \cdots \hat{\text{ओ}}_n(x_n) \right\rangle_e = \int \mathcal{D}\text{ऽ}_i \, \hat{\text{ओ}}_1(x_1) \cdots \hat{\text{ओ}}_n(x_n) \, \mathrm{e}^{-S[\text{ऽ}_i, e^a_\mu]} \tag{5.18}$$

For the rest of this derivation, we write $\hat{\text{ओ}}_1(x_1) \cdots \hat{\text{ओ}}_n(x_n) \coloneqq X$ to make the rest of the expressions simpler to write. The inifinitesimal reparametrisation $x^\mu \mapsto x^\mu + \xi^\mu(x^\nu)$ gives rise to the variations:

$$\delta e^a_\mu = -\left(\partial_\nu e^a_\mu \right) \xi^\nu - \left(\partial_\mu \xi^\nu \right) e^a_\nu, \quad \delta\text{ऽ} = -\xi^\mu \, \partial_\mu \text{ऽ} \tag{5.19}$$

Meanwhile, the action and measure are invariant under the reparametrisation. We can apply these reparametrisation properties to $Z \left\langle \hat{\text{ओ}}_1(x_1) \cdots \hat{\text{ओ}}_n(x_n) \right\rangle_e =: Z \left\langle X \right\rangle_e$ to get an expression for $Z \left\langle X + \delta X \right\rangle_{e+\delta e}$, and from this an expression for $\left\langle X + \delta X \right\rangle_{e+\delta e}$:

$$\begin{aligned} Z \left\langle X + \delta X \right\rangle_{e+\delta e} &= \int \left[\mathcal{D}\text{ऽ}_i + \mathcal{D}(\delta\text{ऽ}_i) \right]_{e+\delta e} (X + \delta X) \, \mathrm{e}^{-S[\text{ऽ}_i + \delta\text{ऽ}_i, \, e+\delta e]} \\ &= \int \left[\mathcal{D}\text{ऽ}_i \right]_e (X + \delta X) \, \mathrm{e}^{-S[\text{ऽ}_i, e]} = Z_e \left\langle X \right\rangle_e + Z_e \left\langle \delta X \right\rangle_e \end{aligned} \tag{5.20}$$

Taking $X = \mathbb{1}$, we have $Z_{e+\delta e} = Z_e$: the vacuum partition function is invariant under reparametrisations. This then gives $\left\langle X + \delta X \right\rangle_{e+\delta e} = \left\langle X \right\rangle_e + \left\langle \delta X \right\rangle_e$. Meanwhile, we can apply a change of variables for the

integral from $\mathfrak{ξ}_i + \delta\mathfrak{ξ}_i$ to $\mathfrak{ξ}_i$ and apply (5.4) to get an equivalent expression for $Z\langle X + \delta X\rangle_{e+\delta e}$:

$$\begin{aligned} Z\langle X+\delta X\rangle_{e+\delta e} &= \int [\mathcal{D}\mathfrak{ξ}_i]_{e+\delta e}\, X\, \mathrm{e}^{-S[\mathfrak{ξ}_i,\, e+\delta e]} \\ &= \int [\mathcal{D}\mathfrak{ξ}_i]_{e+\delta e} \left[X\, \mathrm{e}^{-S[\mathfrak{ξ}_i,\, e]}\Big(1 + \int \mathrm{d}^d x\, \det e^a_\nu\, \delta e^a_\mu\, \langle T^\mu_a X\rangle\Big)\right] \\ &= Z_e\langle X\rangle_e + Z_e \int \mathrm{d}^d x\, \det e^a_\nu\, \delta e^a_\mu\, \langle T^\mu_a X\rangle \end{aligned} \tag{5.21}$$

Combining this with the expression $\langle X+\delta X\rangle_{e+\delta e} = \langle X\rangle_e + \langle \delta X\rangle_e$ derived from (5.20), we have:

$$\langle \delta X\rangle_e = \int \mathrm{d}^d x\, \det e^a_\nu\, \delta e^a_\mu\, \langle T^\mu_a X\rangle_e \tag{5.22}$$

Applying the variation $(\partial_\nu e^a_\nu)\,\xi^\nu - (\partial_\mu \xi^\nu)\, e^a_\nu$ from (5.19) and applying the flat space condition from (1.1), we have each side of this as:

$$\begin{aligned} \langle \delta X\rangle_e &= -\sum_i \xi^\nu(x^\rho)\frac{\partial\langle X\rangle}{\partial x^\nu} \\ \int \mathrm{d}^d x\, \det e^a_\nu\; \delta e^a_\mu\, \langle T^\mu_a X\rangle_e &= -\int \mathrm{d}^d x\, \partial_\mu \xi^\nu\, \langle T^\mu_\nu X\rangle = \int \mathrm{d}^d x\, \xi^\nu \partial_\mu\, \langle T^\mu_\nu X\rangle \end{aligned} \tag{5.23}$$

Since this is an arbitrary reparametrisation, this gives us the conformal Ward identity for translation invariance:

$$\frac{\partial\left\langle T^\mu_\nu\, \hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)\right\rangle}{\partial x^\mu} = -\sum_{i=1}^n \delta(x-x_i)\frac{\partial\left\langle \hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)\right\rangle}{\partial x^\nu_i} \tag{5.24}$$

(Here, I simply replaced $X \coloneqq \hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)$ at the end.) The derivation for rotational invariance is exactly the same as for translational invariance, except here we instead apply the infinitesimal rotation $x^\mu \mapsto \omega^\mu_\nu\, x^\nu = g_{\alpha\nu}\,\omega^\mu_\nu\, x^\nu$. This infinitesimal rotation gives rise to the vielbein and field infinitesimal rotations, and the respective variations:

$$\begin{aligned} e^a_\mu &\mapsto e^a_\mu + \omega^{ab}(x^\nu)\, e_{b\mu}, \quad \mathfrak{ξ}_i \mapsto \mathfrak{ξ}_i - \frac{\mathrm{i}}{2}\omega^{ab}(x^\nu)\, S_{abi}\, \mathfrak{ξ}_i; \\ \delta e^a_\nu &= -\varepsilon^a_c\, \omega^{cb}\, e_{b\mu} - \varepsilon^c_b\, \omega^{ac}\, e_{b\mu} - \varepsilon^a_c\, e^c_\nu\big(\partial^\nu\, e_{b\mu}\big), \quad \delta\mathfrak{ξ}_i = \omega^{ab}\, S_{abi}\, \mathfrak{ξ}_i \end{aligned} \tag{5.25}$$

Applying the same steps as (5.18)–(5.24), we get the conformal Ward identity for rotation invariance:

$$\begin{aligned} \varepsilon^{\mu\nu}\left\langle T^{\mu\nu}(x)\Big(\hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)\Big)\right\rangle &= -\mathrm{i}\sum_{i=1}^n \delta(x-x_i)\, S_{\mu\nu i}\left\langle \hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)\right\rangle \\ &= \left\langle \big(T_{\mu\nu}(x) - T_{\nu\mu}(x)\big)\Big(\hat{\text{ϑ}}_1(x_1)\cdots\hat{\text{ϑ}}_n(x_n)\Big)\right\rangle \end{aligned} \tag{5.26}$$

5.4 Conformal Ward Identity for Scale Invariance

For the Ward identity for scale invariance, we switch tactics and examine the case of 2D explicitly. Quoting [5] directly[10]:

> It is here that we must distinguish the case of two dimensions from the others. In three or more dimensions an action cannot be invariant under a local scale transformation: The use of tetrads and covariant derivatives allows us to define actions invariant under local rotations of the frames, but not under local scalings. In contrast, the two-dimensional conformal group includes local scale transformations and we may proceed as before, and end up with the [conformal] Ward identity [for scale invariance for primary operators].

As such, we follow the derivation in [1, 2], and provide the expression in [5] at the end for reference. Considering the contour integral of $\epsilon(z)\,T(z)$ inserted into the n-point function $\langle \hat{\mathfrak{q}}_1(x_1)\cdots\hat{\mathfrak{q}}_n(x_n)\rangle$ of *primaries*, and applying (5.11), we have:

$$\begin{aligned}
&\left\langle \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\epsilon(z)\,T(z)\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1,\bar{\mathfrak{z}}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n,\bar{\mathfrak{z}}_n)\right\rangle \\
&\quad= \sum_{i=1}^{n}\left\langle \hat{\mathfrak{q}}_1(\mathfrak{z}_1,\bar{\mathfrak{z}}_1)\cdots\left(\oint_{\mathcal{C}(\mathfrak{z}_1)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\epsilon(z)\,T(z)\,\hat{\mathfrak{q}}_i(\mathfrak{z}_i,\bar{\mathfrak{z}}_i)\right)\cdots\hat{\mathfrak{oi}}_n(\mathfrak{z}_n,\bar{\mathfrak{z}}_n)\right\rangle \\
&\quad= \sum_{i=1}^{n}\oint_{\mathcal{C}(\mathfrak{z}_1)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\epsilon(z)\left\langle \hat{\mathfrak{q}}_1(\mathfrak{z}_1,\bar{\mathfrak{z}}_1)\cdots\left(\frac{\Delta_{\mathfrak{q}}\,\hat{\mathfrak{q}}}{(z-\mathfrak{z})^2}+\frac{1}{z-\mathfrak{z}}\frac{\partial\hat{\mathfrak{q}}}{\partial\mathfrak{z}}+f_{\text{reg}}(z)\right)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n,\bar{\mathfrak{z}}_n)\right\rangle = 0
\end{aligned} \tag{5.27}$$

Here, we used the deformation of the contour integrals in Figure 12, modified from [1]. This holds for *all* $\epsilon(z)$; thus, we can pick $\epsilon(z) = -z^{n+1}$ to set this expansion equal to an expansion in the Virasoro generators, or equivalently an expansion in $T(z)$ itself (given (5.5)). This gives the *integrand* as zero, making this expression valid identically. This gives the conformal Ward identity for scale invariance for primary operators:

$$\left\langle T(z)\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1,\bar{\mathfrak{z}}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n,\bar{\mathfrak{z}}_n)\right\rangle = \sum_{i=1}^{n}\left(\frac{\Delta_{\mathfrak{q}}\,\hat{\mathfrak{q}}}{(z-\mathfrak{z})^2}+\frac{1}{z-\mathfrak{z}}\frac{\partial\hat{\mathfrak{q}}}{\partial\mathfrak{z}}\right)\left\langle \hat{\mathfrak{q}}_1(\mathfrak{z}_1,\bar{\mathfrak{z}}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n,\bar{\mathfrak{z}}_n)\right\rangle \tag{5.28}$$

Just for completeness, we briefly discuss the derivation using vielbeins, as provided in [5]. Again, we can only restrict ourselves to primary operators, since these correspond to states $|\mathfrak{q}\rangle$ of well-defined scalings $\Delta_{\mathfrak{q}}$ (as discussed in §3.1)[11]. For these, the infinitesimal transformations $e^a_\mu \mapsto e^a_\mu + \epsilon(x)\,e^a_\mu$ and $\mathfrak{q}_i \mapsto \mathfrak{q}_i - \epsilon(x)\,\Delta_i\mathfrak{q}_i$ yield (using the same process as (5.18)–(5.24)) the conformal Ward identity for scale invariance as:

$$\left\langle T^\mu_\mu(z)\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\right\rangle = \sum_{i=1}^{n}\delta(z-\mathfrak{z}_i)\,\Delta_i\left\langle \hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\right\rangle \tag{5.29}$$

[10] Again, I don't think I can put it any more succinctly than the original source already does.
[11] gain quoting directly from [5], since they express it quite succinctly already, "Since we are performing an arbitrary local scaling, only primary fields (as opposed to quasi-primary) will transform as above."

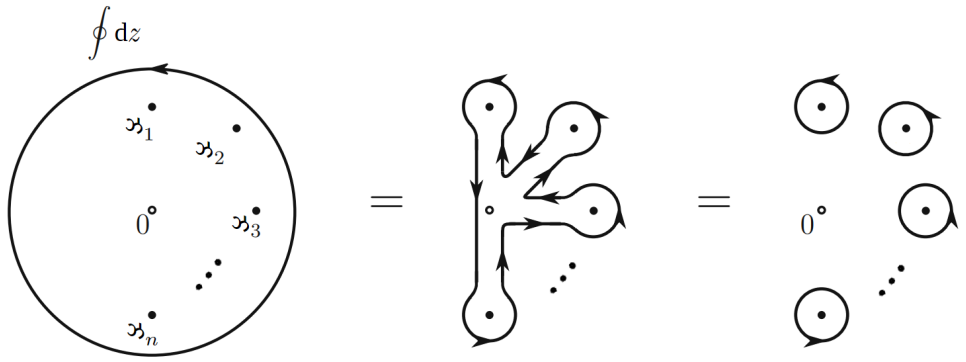

Figure 12: Deformation of the contour used in (5.27) to retrieve the conformal Ward identity for scale invariance. Modified from [1].

Finally, we apply the specific conformal transformations $\epsilon(z) = 1$, $\epsilon(z) = z$, and $\epsilon(z) = z^2$ to get the global conformal Ward identities [26] for 2D, which gives us the conformal Ward identities in terms of the Virasoro generators. In terms of these conformal transformations, the global conformal Ward identities are given by:

$$\sum_{i=1}^{n} \left\langle L_{-1}(z_i)\, \hat{\mathfrak{q}}_1(\mathfrak{z}_1, \bar{\mathfrak{z}}_1) \cdots \hat{\mathfrak{q}}_n(\mathfrak{z}_n, \bar{\mathfrak{z}}_n) \right\rangle = 0 \tag{5.30}$$

$$\sum_{i=1}^{n} \left\langle (L_0(z_i) + z_i L_{-1}(z_i))\, \hat{\mathfrak{q}}_1(\mathfrak{z}_1, \bar{\mathfrak{z}}_1) \cdots \hat{\mathfrak{q}}_n(\mathfrak{z}_n, \bar{\mathfrak{z}}_n) \right\rangle = 0 \tag{5.31}$$

$$\sum_{i=1}^{n} \left\langle (L_1(z_i) + 2z_i L_0(z_i) + z_i^2 L_{-1}(z_i))\, \hat{\mathfrak{q}}_1(\mathfrak{z}_1, \bar{\mathfrak{z}}_1) \cdots \hat{\mathfrak{q}}_n(\mathfrak{z}_n, \bar{\mathfrak{z}}_n) \right\rangle = 0 \tag{5.32}$$

6 Operator Product Expansion and Conformal Blocks

6.1 Operator Product Expansion in Perturbative QFT

As discussed in §2.3, correlation functions of operators beyond the three-point functions are far less constrained by conformal invariance, due to the presence of the cross-ratios (2.22); thus, very little can be said about them by applying conformal invariance directly. However, we can use the operator product expansion (OPE) to reduce n-point functions to sums of $(n-1)$-point functions. The OPE idea originally comes from perturbative QFT; thus, we start by introducing the OPE in perturbative QFT, before moving onto the OPE in CFTs. Here, we follow [37].

The OPE provides a way to deal with singular expressions when examining the products of operators. The core observation [38, 39] of the OPE is that if a product of operators $\hat{\mathfrak{m}}(y^\mu)\, \hat{\mathfrak{o}}(x^\mu)$ is analytic in $(y^\mu - x^\mu)$, then its Fourier transform $\hat{\mathfrak{m}}(k_\mu)\, \hat{\mathfrak{o}}(0) = \mathcal{F}\{\hat{\mathfrak{m}}(y^\mu)\, \hat{\mathfrak{o}}(x^\mu)\}$ will decrease exponentially as we take the Fourier conjugate variable $k^\mu \to \infty$; conversely, the singularities in $\hat{\mathfrak{m}}(y^\mu)\, \hat{\mathfrak{o}}(x^\mu)$ as y^μ approaches x^μ correspond to the leading terms in the $k^\mu \to \infty$ limit of $\hat{\mathfrak{m}}(k_\mu)\, \hat{\mathfrak{o}}(0)$.

The claim of the OPE is that as $y^\mu \to x^\mu$, the singular part of $\hat{\mathfrak{m}}(y^\mu)\, \hat{\mathfrak{o}}(x^\mu)$ is given by a sum over *other*,

purely local, operators $\hat{\text{थ}}$ located at x^μ:

$$\lim_{y^\mu \to x^\mu} \hat{\text{म}}(y^\mu)\, \hat{\text{छ}}(x^\mu) = \sum_{\text{थ}} f_{\text{मछथ}}(y^\mu - x^\mu)\, \hat{\text{थ}} \tag{6.1}$$

Here, $\{f_{\text{मछथ}}\}$ are singular c-number functions which together encode all of the singularity properties of $\hat{\text{म}}(y^\mu)\, \hat{\text{छ}}(x^\mu)$ in the $y^\mu \to x^\mu$ limit. We can see the logic of this claim by examining the Fourier transform of $\hat{\text{म}}(y^\mu)\, \hat{\text{छ}}(x^\mu)$. The Fourier transform of (6.1), including the necessary time ordering, is given by:

$$\lim_{k \to 0} \int \mathrm{d}^d y\; \mathcal{T}\left\{ \mathrm{e}^{-\mathrm{i} k_\mu y^\mu}\, \hat{\text{म}}(y^\mu)\, \hat{\text{छ}}(x^\mu) \right\} = \sum_{\text{थ}} g_{\text{मछथ}}\, \hat{\text{थ}}\big(k_\mu = 0\big) \tag{6.2}$$

Here, $\{g_{\text{मछथ}}(0)\}$ are functions of k_μ corresponding to the Fourier transforms of the coefficients $f_{\text{मछथ}}$. A generic operator $\hat{\text{ओ}}$ will be constructed out of the fields $\{\Phi_a\}$ that make up the theory, as well as their nth order derivatives $\{\partial_i \cdots \partial_j \Phi_a\}$. As $\hat{\text{ओ}}$ gets more and more complex (i.e., is constructed out of more and more products of fields and field derivatives), the strength of the singularity in $f_{\text{मछथ}}$ decreases. As a result, as $\hat{\text{ओ}}$ gets more complex, $g_{\text{मछथ}}$ decreases increasingly rapidly in the $k \to \infty$ limit.

We examine the presence of the local operators $\hat{\text{म}}(y_1)$, $\hat{\text{छ}}(y_2)$, $\hat{\text{न}}_1(x_1)$, and $\hat{\text{न}}_2(x_2)$ in an overall vacuum connected Green function, where the expansions of these operators in terms of $\{\Phi_a\}$ and $\{\partial_i \cdots \partial_j \Phi_a\}$ are written respectively as $\text{ꠢ}(y_1)$, $\text{ꠚ}(y_2)$, $\text{ꠝ}_1(x_1)$, and $\text{ꠝ}_2(x_2)$. Examining this Green function where y_1 and y_2 approach the point x, where and $\{x_i\}$ (including x_1 and x_2) are a bunch of fixed points that are much further away from y_1, y_2, and x than y_1, y_2, and x are from each other; we have:

$$\begin{aligned} \left\langle \mathcal{T}\left\{ \cdots \hat{\text{म}}(y_1)\, \hat{\text{छ}}(y_2) \cdots \hat{\text{न}}_1(x_1)\, \hat{\text{न}}_2(x_2) \cdots \right\} \right\rangle_0 &= \int \prod_{a,\, z} \mathcal{D}\Phi_a(z) \left[\cdots \hat{\text{म}}(y_1)\, \hat{\text{छ}}(y_2) \cdots \hat{\text{न}}_1(x_1)\, \hat{\text{न}}_2(x_2) \cdots \mathrm{e}^{\mathrm{i} S[\Phi_a]} \right] \\ &= \int \prod_{a,\, z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ꠢ}(y_1)\, \text{ꠚ}(y_2) \cdots \text{ꠝ}_1(x_1)\, \text{ꠝ}_2(x_2) \cdots \mathrm{e}^{\mathrm{i} S[\Phi_a]} \right] \end{aligned} \tag{6.3}$$

From this expression, we now construct a ball $B(\text{२})$ of radius ꠲ around y_1 and y_2, such that $|\text{२}| \gg |y_1 - y_2|$ but $|\text{२}| \ll |x_i - x_j|$ for all of the x_is (including x_1 and x_2, but excluding x); this is depicted in Figure 13. Since the action $\int \mathrm{d}^d z\, \mathcal{L}(z)$ is entirely local, we can separate the action to the region within the ball and the region excluding it:

$$S = \int \mathrm{d}^d z\, \mathcal{L}(z) = \int\limits_{z \in B(\text{२})} \mathrm{d}^d z\, \mathcal{L}(z) + \int\limits_{z \notin B(\text{२})} \mathrm{d}^d z\, \mathcal{L}(z) \tag{6.4}$$

This lets us factorise $\left\langle \mathcal{T}\{\cdots \hat{\text{ম}}(y_1)\,\hat{\text{ছ}}(y_2)\cdots \hat{\text{ন}}_1(x_1)\,\hat{\text{ন}}_2(x_2)\cdots\}\right\rangle_0$ accordingly:

$$\begin{aligned}&\left\langle \mathcal{T}\{\cdots \hat{\text{ম}}(y_1)\,\hat{\text{ছ}}(y_2)\cdots \hat{\text{ন}}_1(x_1)\,\hat{\text{ন}}_2(x_2)\cdots\}\right\rangle_0 \\ &\qquad = \left(\int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ম}(y_1)\,\text{ছ}(y_2)\cdots \exp\left\{\text{i}\int\limits_{z\,\in\,B(\text{২})} S[\Phi_a]\right\}\right]\right)\cdot \\ &\qquad \left(\int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ন}_1(x_1)\,\text{ন}_2(x_2)\cdots \exp\left\{\text{i}\int\limits_{z\,\notin\,B(\text{২})} S[\Phi_a]\right\}\right]\right)\end{aligned} \tag{6.5}$$

The overall path integral (inside and outside the ball put together) is taken over the space of C^∞ functions on the manifold the fields live on [40]; thus, the path integral over $\mathcal{D}\Phi_a$ inside $B(\text{২})$ is constrained by the boundary condition that the fields inside $B(\text{২})$ must merge smoothly (to all derivative orders) with the fields outside $B(\text{২})$. Aside from this condition, however, the path integral inside $B(\text{২})$ is completely distinct from the path integral outside $B(\text{২})$.

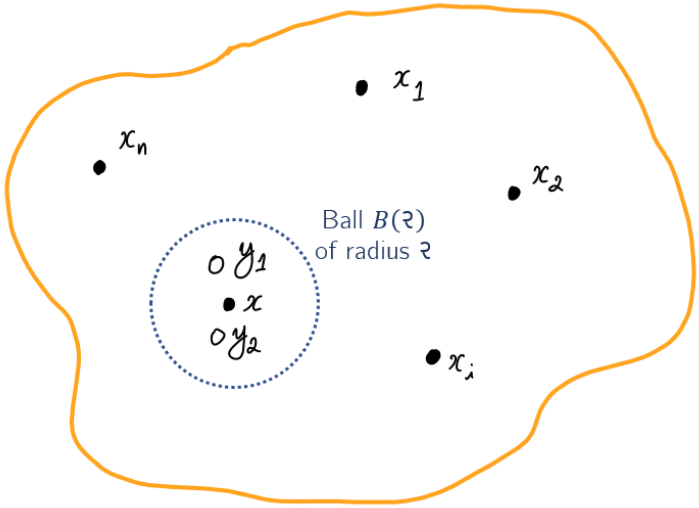

Figure 13: Constructing a ball $B(\text{২})$ of radius ২ around the points y_1, y_2, and x; such that it includes these three points and excludes every other point that other operators are valued at.

As a result, we can express the path integral *inside* $B(\text{২})$ in terms of the values of the fields and the field derivatives on the *surface*, which, due to the boundary conditions, we can then express in terms of the fields and the field derivatives *outside* $B(\text{২})$ with their values extrapolated to x. If we define $\text{থ}(x)$ as the product of fields and field derivatives outside of $B(\text{২})$ extrapolated to x, then we can express the path integral inside $B(\text{২})$ in a Taylor-Madhava series expansion in থ at x:

$$\int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ম}(y_1)\,\text{ছ}(y_2)\cdots \exp\left\{\text{i}\int\limits_{z\,\in\,B(\text{২})} S[\Phi_a]\right\}\right] = \sum_{\text{থ}} C_{\text{মছথ}}(\{y_{\text{থ}}\}-x)\,\text{থ}(x) \tag{6.6}$$

Here, $C_{\text{মছথ}}$ are the coefficients of the Taylor-Madhava expansion in থ at x. These coefficients can only

be functions of the coordinate differences within the ball (i.e., $C_{\text{ϻϭϑ}}$ is a function of $y_1 - x$, $y_2 - x$, etc.). Since the points $\{x_i\}$ are far outside $B(\text{ϩ})$, excluding $B(\text{ϩ})$ from the action (i.e., expressing the action as $S = \int_{z \notin B(\text{ϩ})} \mathrm{d}^d z \, \mathcal{L}(z)$) has no effect on the correlation function in the limit as $\text{ϩ} \to 0$. Taking this limit, we have (6.5) given by:

$$\begin{aligned}
&\lim_{\text{ϩ}\to 0} \left\langle \mathcal{T}\left\{ \cdots \hat{\text{ϻ}}(y_1)\, \hat{\text{ϭϑ}}(y_2) \cdots \hat{\text{ϩ}}_1(x_1)\, \hat{\text{ϩ}}_2(x_2) \cdots \right\} \right\rangle_0 \\
&\qquad = \left(\int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ϩ}_1(x_1)\, \text{ϩ}_2(x_2) \cdots \exp\left\{ \mathrm{i} \int_{z \notin B(\text{ϩ})} S[\Phi_a] \right\} \right] \right) \left(\sum_{\text{ϑ}} C_{\text{ϻϭϑ}}(\{y_{\text{ϑ}}\} - x)\, \text{ϑ}(x) \right) \\
&\qquad\qquad = \sum_{\text{ϑ}} C_{\text{ϻϭϑ}}(\{y_{\text{ϑ}}\} - x) \int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ϩ}_1(x_1)\, \text{ϩ}_2(x_2) \cdots \text{ϑ}(x) \exp\left\{ \mathrm{i} \int_{z \notin B(\text{ϩ})} S[\Phi_a] \right\} \right]
\end{aligned} \tag{6.7}$$

As mentioned when writing (6.3), the expressions $\{\text{ϩ}_\ell(x_\ell)\}$ referred to the specific products of fields and field derivatives that corresponded to the operators $\hat{\text{ϩ}}$ at x_ℓ. Similarly, the expression $\text{ϑ}(x)$ corresponds to the specific product of fields and field derivatives expressing the local operator $\hat{\text{ϑ}}$ at x, so replacing $\{\text{ϩ}_\ell(x_\ell), \text{ϑ}(x)\}$ with $\left\{\hat{\text{ϩ}}_\ell(x_\ell), \hat{\text{ϑ}}(x)\right\}$ in (6.7) finally gives us the OPE (expressed in terms of the vacuum connected correlation function):

$$\begin{aligned}
&\left\langle \mathcal{T}\left\{ \cdots \hat{\text{ϻ}}(y_1)\, \hat{\text{ϭϑ}}(y_2) \cdots \hat{\text{ϩ}}_1(x_1)\, \hat{\text{ϩ}}_2(x_2) \cdots \right\} \right\rangle_0 \\
&\qquad = \sum_{\text{ϑ}} C_{\text{ϻϭϑ}}(\{y_{\text{ϑ}}\} - x) \int \prod_{a,\,z} \mathcal{D}\Phi_a(z) \left[\cdots \text{ϩ}_1(x_1)\, \text{ϩ}_2(x_2) \cdots \text{ϑ}(x) \exp\left\{ \mathrm{i} \int_{z \notin B(\text{ϩ})} S[\Phi_a] \right\} \right] \\
&\qquad = \sum_{\text{ϑ}} C_{\text{ϻϭϑ}}(\{y_{\text{ϑ}}\} - x) \left\langle \mathcal{T}\left\{ \cdots \hat{\text{ϑ}}(x) \cdots \hat{\text{ϩ}}_1(x_1)\, \hat{\text{ϩ}}_2(x_2) \cdots \right\} \right\rangle_0
\end{aligned} \tag{6.8}$$

Taking $\left\{\hat{\text{ϩ}}_\ell(x_\ell)\right\} = \mathbb{1}$ and adding the spin indices a, b, and c for various (maybe different) representations of $\mathsf{SO}(d)$ finally gives us the direct version of the OPE:

$$\mathcal{T}\left\{ \hat{\text{ϻ}}_a(y_1)\, \hat{\text{ϭ}}_b(y_2) \right\} = \sum_{\text{ϑ}} C^{abc}_{\text{ϻϭϑ}}(\{y_{\text{ϑ}}\} - x)\, \hat{\text{ϑ}}_c(x) \tag{6.9}$$

6.2 Conformal OPE

The expression of the OPE just derived in §6.1 relied solely on the locality of the operators and the fact that the fields are C^∞ functions. Notably, this *didn't* rely on the specific symmetry group the fields lived in; as such, (6.9) is exactly as valid for CFTs, with no modifications at all. Instead, we can use the OPE to significantly constrain n-point correlation functions for $n \geq 4$.

Before continuing, we note that the Ward identities in §5 were all, themselves, specific examples of OPEs.

In §2, we saw that the four-point function was far less constrained by conformal invariance than the two- and three-point functions, due to the emergence of the cross ratios u and v. This problem persists for

all $n \geq 4$, meaning that conformal invariance could tell us very little directly about these correlation functions. However, the OPE (6.9) allows us to express the n-point function as a sum over $(n-1)$-point functions, thereby allowing us to whittle expressions down until we get to one of the expressions in §2.

The first example of using the OPE to express the n-point function as a sum over $(n-1)$-point functions starts using (6.9) itself; in §2.1, we had conformal invariance constraining the form of the two-point function for two *general* operators to be of the form (2.4). As a result, we have (6.9) given by:

$$\left\langle \mathcal{R}\left\{ \hat{\text{ओ}}_{ia}(y_i)\, \hat{\text{ओ}}_{jb}(y_j) \right\} \right\rangle = \sum_{\ell} C_{ij\ell}^{abc}(\{y_\ell\} - x) \left\langle \hat{\text{ओ}}_{\ell c}(x) \right\rangle = \frac{\hbar}{|y_i - y_j|^{\Delta_i + \Delta_j}} \tag{6.10}$$

From §3.1, we were able to identify operators with states in the generalised Verma module, with the spectral decomposition given by (3.8) (or, equivalently, by (3.9)). As such, we need to evaluate the expectation values in (6.10) in the eigenbasis of the generalised Verma module. Applying (3.8)–(3.9) to (6.10), we see the final expression in (6.10) can only be valid in *general* if $C_{ij\ell}^{abc}$ has the form given by:

$$\begin{aligned} C_{ij\ell}^{abc} &= \frac{\hbar}{|y_i - y_j|^{\Delta_i + \Delta_j - \Delta_m}} \sum_{\ell, m, n} \prod_{\{m,n\}} c_{\ell mn} x^{\mu_m} \partial_{\mu_n} \\ &= \hbar |y_1 - y_2|^{\Delta_m - \Delta_i - \Delta_j} \left(1 + c_1 x^\mu \partial_\mu + c_2\, x^\mu x^\nu \partial_\mu \partial_\nu + c_3\, x^2 \partial^2 + \cdots\right) \end{aligned} \tag{6.11}$$

We derive this expression for the specific case of spinless primaries below. In general, the coefficients $\{c_{\ell mn}\}$ depend on the specific form of $\hat{\text{ओ}}_{ia}(y_i)$ and $\hat{\text{ओ}}_{jb}(y_j)$; and depend specifically on Δ_i and Δ_j. These coefficients can be determined by multiplying both sides of (6.9) with a third operator $\hat{\text{ओ}}_{md}$ and taking the expectation value of the entire expression:

$$\left\langle \hat{\text{ओ}}_{ia}(y_i)\, \hat{\text{ओ}}_{jb}(y_j)\, \hat{\text{ओ}}_{md}(y_m) \right\rangle = \sum_{\ell} C_{ij\ell}^{abc}(\{y_\ell\} - x) \left\langle \hat{\text{ओ}}_{\ell c}(x)\, \hat{\text{ओ}}_{md}(y_m) \right\rangle \tag{6.12}$$

In general, this process generalises: we can employ the same procedure to reduce the n-point function to an $(n-1)$-point function, and perform this iteratively until we get a correlator in one of the forms provided in §2. The general expression of (6.12) is given by:

$$\left\langle \hat{\text{ओ}}_{ia}(y_i)\, \hat{\text{ओ}}_{jb}(y_j)\, \hat{\text{ओ}}_{md}(y_m) \cdots \hat{\text{ओ}}_{ns}(y_n) \right\rangle = \sum_{\ell} C_{ij\ell}^{abc}(\{y_\ell\} - x) \left\langle \hat{\text{ओ}}_{\ell c}(x)\, \hat{\text{ओ}}_{md}(y_m) \cdots \hat{\text{ओ}}_{ns}(y_n) \right\rangle \tag{6.13}$$

Instead of stopping at one of the expressions in §2, if we continue this derivation all the way down to the one-point functions (i.e., the expectation values $\left\langle \hat{\text{ओ}}_{pf} \right\rangle$ of individual operators $\hat{\text{ओ}}_{pf}$, we note that these are determined by dimensional analysis as:

$$\left\langle \hat{\text{ओ}}_{pf} \right\rangle = \begin{cases} 1 & \hat{\text{ओ}}_{pf} = \mathbb{1}, \text{ CFT on flat space manifold} \\ 0 & \hat{\text{ओ}}_{pf} \neq \mathbb{1}, \text{ CFT on flat space manifold} \\ \beta^{-\Delta_{\text{ओ}}} & \hat{\text{ओ}}_{pf} \neq \mathbb{1}, \text{ CFT on nontrivial manifold (equivalent to finite-temp. CFT)} \end{cases} \tag{6.14}$$

An important consequence of this is that *every* n-point correlation function is determined by the conformal dimensions $\{\Delta_n\}$, the spins $\{s_n\}$, and the OPE coefficients $\left\{C_{ij\ell}^{abc}\right\}$. This set of information taken together is the conformal data. In addition to the reduction procedure for the n-point function, we can extract a Ward identity for the OPE as well; here, we follow [26]. Starting with the general expression (6.9) and restricting ourselves to primary operators, we can insert a factor of $(z-y_j)^{n+1}\,T(z)$ on each side, and integrate each expression for $n \geq -1$ over $\mathrm{d}z$ over a contour $\mathcal{C}$ encircling both y_i and y_j:

$$\oint_{\mathcal{C}} \mathrm{d}z\,(z-y_2)^{n+1}\,\mathcal{R}\{T(z)\,\hat{\mathfrak{P}}_{ia}(y_i)\,\hat{\mathfrak{P}}_{jb}(y_j)\} = \sum_k C_{ijk}^{abc}(\{y_k\}-x)\,\hat{\mathfrak{P}}_{ck}(x) \tag{6.15}$$

Applying the OPE (5.11) between the stress-energy tensor and primary operators, this gives the operator product expansion Ward identity:

$$\left(L_n(y_j) + \sum_{m=-1}^{n}\binom{m+1}{n+1}|y_i-y_j|^{n-m}\,L_m(y_i)\right)\hat{\mathfrak{P}}_{ia}(y_i)\,\hat{\mathfrak{P}}_{jb}(y_j) = \sum_k C_{ijk}^{abc}\,L_n\,\hat{\mathfrak{P}}_{kc} \tag{6.16}$$

To give an explicit example of the n-point reduction procedure, we consider spinless primary operators in (6.12) (replacing the labelling of $\hat{\text{ओ}}_{ia}$, $\hat{\text{ओ}}_{jb}$, and $\hat{\text{ओ}}_{md}$ with the primary operators $\hat{\mathfrak{P}}_i$, $\hat{\mathfrak{P}}_j$, and $\hat{\mathfrak{P}}_m$ and drop the spin labels on $C_{ij\ell}^{abc}$). Then, the left-hand side is given by (2.17)–(2.18). For the right-hand side, we can orthonormalise the generalised Verma module by normalising (2.13):

$$\langle\hat{\mathfrak{P}}_\ell(x)\,\hat{\mathfrak{P}}_m(y_m)\rangle = \frac{\delta_{\ell m}\,\delta_{s_\ell s_m}}{|x-y_\ell|^{2\Delta_\ell}}\left(\partial_{(\nu_1}\hat{\jmath}^{(\mu_1}\cdots\partial_{\nu_s)}\hat{\jmath}^{\mu_s)} - \prod_{n,m=1}^{s}\delta_{\mu_n\cdots\mu_m}\right) \tag{6.17}$$

Using this, the sum over ℓ condenses to a single term, giving the OPE as:

$$\frac{f_{ij\ell}}{|y_i-y_j|^{\Delta_i+\Delta_j-\Delta_\ell}|y_j-y_\ell|^{\Delta_j+\Delta_\ell-\Delta_i}|y_i-y_k|^{\Delta_i+\Delta_\ell-\Delta_j}} = \frac{C_{ij\ell}}{|y_j-y_\ell|^{-2\Delta_\ell}} \tag{6.18}$$

Thus, the OPE coefficient $C_{ij\ell}$ is given by $f_{ij\ell}$ times a differential operator $\hat{\mathbb{D}}_{ijk}$ dependent *solely* on Δ_i, Δ_j, Δ_k. Then, by expanding both sides of (6.18) around $|y_i-y_j|/|y_j-y_\ell| = 0$ and equating them term-by-term, we can determine the $\{c_{\ell mn}\}$s in (6.11). For the specific case of $\Delta_j = \Delta_i$, this gives $C_{ij\ell}$ as:

$$\begin{aligned} C_{ii\ell} = f_{ii\ell}\,|y_i-y_\ell|^{\Delta_\ell-2\Delta_i} \\ \cdot\left(1+\frac{1}{2}x^\mu\partial_\mu + \frac{\Delta_\ell}{8(\Delta_\ell+1)}x^\mu x^\nu\partial_\mu\partial_\nu - \frac{\Delta_\ell}{8(2\Delta_\ell-d-2)(\Delta_\ell+1)}x^2\partial^2+\cdots\right) \end{aligned} \tag{6.19}$$

Overall, this gives us the OPE of two identical scalar primaries as:

$$\hat{\mathfrak{P}}(y_1)\,\hat{\mathfrak{P}}(y_2) = \sum_{\text{ब}} f_{\text{पपओ}}\,C_{\mu_1\cdots\mu_s}(x_1-x_2)\,\hat{\text{ओ}}^{\mu_1\cdots\mu_s}(x_2) \tag{6.20}$$

Before continuing, we note that when examining the OPE of two scalar operators (in general, not just

scalar primaries), we require the operators in the expansion to be traceless, symmetric, and be of even spin.

6.3 Conformal Blocks

Again considering the case of spinless primaries, we had the four-point function given by (2.23). Considering identical scalar primaries, we can apply (2.23) to get the simplified expression:

$$\left\langle \hat{\text{पे}}(x_i)\,\hat{\text{पे}}(x_j)\,\hat{\text{पे}}(x_k)\,\hat{\text{पे}}(x_m)\right\rangle = \prod_{i<j}\frac{g(u,v)}{\left|x_i-x_j\right|^{2\Delta_{\text{ब}}}} \tag{6.21}$$

On the other hand, we can pair up the primaries and expanding them in OPEs:

$$\left\langle \overbrace{\hat{\text{पे}}(x_i)\,\hat{\text{पे}}(x_j)}\,\overbrace{\hat{\text{पे}}(x_k)\,\hat{\text{पे}}(x_m)}\right\rangle = \sum_{\text{ओ},\,\text{ज}} f_{\text{पपओ}}\,f_{\text{पपज}}\,C^a_{\text{ओ}}(x_i-x_j)\,C^b_{\text{ज}}(x_k-x_m)\left\langle \hat{\text{ओ}}^a(x_k)\,\hat{\text{ज}}^b(x_m)\right\rangle \tag{6.22}$$

Applying the orthonormal basis (6.17) for the generalised Verma module, this gives the four-point correlator as:

$$\left\langle \overbrace{\hat{\text{पे}}(x_i)\,\hat{\text{पे}}(x_j)}\,\overbrace{\hat{\text{पे}}(x_k)\,\hat{\text{पे}}(x_m)}\right\rangle = \sum_{\text{ओ},\,\text{ज}}\frac{f^2_{\text{पपओ}}\,C^a_{\text{ओ}}\,C^b_{\text{ज}}}{\left|x_j-x_m\right|^{2\Delta_{\text{ब}}}}\left(\partial_{(\nu_1}\hat{\jmath}^{(\mu_1}\cdots\partial_{\nu_s)}\hat{\jmath}^{\mu_s)}-\prod_{n,m=1}^{s}\delta_{\mu_n\cdots\mu_m}\right) \tag{6.23}$$

We can define the conformal blocks / conformal partial waves, which are functions of the cross-ratios alone, and allow for a decomposition in terms of the cross-ratios *alone*:

$$C_{\Delta_{\text{ओ}},\,s_{\text{ओ}}} = \frac{\left|x_i-x_j\right|^{\Delta}\left|x_j-x_m\right|^{\Delta_{\text{ब}}}C^a_{\text{ओ}}\,C^b_{\text{ज}}}{\left|x_j-x_m\right|^{2\Delta}}\left(\partial_{(\nu_1}\hat{\jmath}^{(\mu_1}\cdots\partial_{\nu_s)}\hat{\jmath}^{\mu_s)}-\prod_{n,m=1}^{s}\delta_{\mu_n\cdots\mu_m}\right) \tag{6.24}$$

In terms of the conformal blocks, we have $g(u,v)$ given by the conformal block decomposition:

$$g(u,v) = \sum_{\text{ओ}} f^2_{\text{पपओ}}\,C_{\Delta_{\text{ओ}},\,s_{\text{ओ}}}(u,v) \tag{6.25}$$

This gives the conformal block decomposition of the four-point correlator (6.23) as:

$$\left\langle \overbrace{\hat{\text{पे}}(x_i)\,\hat{\text{पे}}(x_j)}\,\overbrace{\hat{\text{पे}}(x_k)\,\hat{\text{पे}}(x_m)}\right\rangle = \sum_{\text{ओ}}\frac{f^2_{\text{पपओ}}\,C_{\Delta_{\text{ओ}},\,s_{\text{ओ}}}(u,v)}{\left|x_i-x_j\right|^{\Delta_{\text{ब}}}\left|x_k-x_m\right|^{\Delta_{\text{ब}}}} \tag{6.26}$$

The conformal block decomposition provides a crucial link between the correlation function as a whole and conformal multiplets (3.6): each individual conformal block represents the contribution of a given conformal multiplet to the four-point function as a whole. The conformal blocks provide a decomposition of n-point functions in terms of the natural basis of the generalised Verma module. In the same way that the geometric symmetry of a problem in E&M or QM provides a natural orthonormal basis for $\mathcal{L}^2$, which we then use to decompose the problem, the conformal blocks are the natural basis for the specific problem at hand, which we can then use to decompose the n-point functions.

Examining the four-point function in radial quantisation, where our coordinate system is chosen so that $|x_k - x_m| \geq |x_i - x_j|$, we have the contracted four-point function in (6.26) given by:

$$\langle \overbrace{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)}\,\overbrace{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)}\rangle = \langle 0|\mathcal{R}\{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)\}\,\mathcal{R}\{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)\}|0\rangle \tag{6.27}$$

Expanding this in the decomposition of the basis given in (3.7) (and relabelling the primaries used in the expansion to $\mathfrak{ब}$ to avoid confusion with the $\hat{\mathfrak{पे}}$s) we have this given by:

$$\begin{aligned}\langle \overbrace{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)}\,\overbrace{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)}\rangle &= \langle 0|\mathcal{R}\{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)\}\,\mathbb{1}\,\mathcal{R}\{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)\}|0\rangle \\ &= \sum_{\mathfrak{ब}} \langle 0|\mathcal{R}\{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)\}\,\hat{\Pi}_{\mathfrak{ब}}\,\mathcal{R}\{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)\}|0\rangle \\ &= \sum_{\mathfrak{ब}} \langle 0|\mathcal{R}\{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)\}|P_{\mu_i}\mathfrak{ब}\rangle\,\langle P_{\mu_j}\mathfrak{ब}|\mathcal{R}\{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)\}|0\rangle\end{aligned} \tag{6.28}$$

Expressing the OPE given in (6.10) in terms of the expression for scalar primaries given in (2.12), we have:

$$\langle \mathcal{R}\{\hat{\mathfrak{पे}}(y_i)\,\hat{\mathfrak{पे}}(y_j)\}\rangle = \sum_{\ell} C_{ij\ell}(\{y_\ell\} - x)\,\langle \hat{\mathfrak{पे}}(x)\rangle = \frac{\mathfrak{k}}{|y_i - y_j|^{2\Delta_{\mathfrak{ब}}}} \tag{6.29}$$

Applying this to (6.28), we have:

$$\langle \overbrace{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)}\,\overbrace{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)}\rangle = \sum_{\mathfrak{ब}} \frac{k_{ij}\,k_{\ell m}}{|x_i - x_j|^{2\Delta_{\mathfrak{प}}}\,|x_k - x_m|^{2\Delta_{\mathfrak{प}}}} \tag{6.30}$$

Applying this to (6.23), we see that the terms in the sum over $\mathfrak{ब}$ correspond to the products of conformal blocks, squared OPE coefficients, and conformal dimensional scalings of the distance magnitudes:

$$\langle \overbrace{\hat{\mathfrak{पे}}(x_i)\,\hat{\mathfrak{पे}}(x_j)}\,\overbrace{\hat{\mathfrak{पे}}(x_k)\,\hat{\mathfrak{पे}}(x_m)}\rangle = \frac{f^2_{\mathfrak{पपब}}\,C_{\mathfrak{ब}}(u,v)}{|x_i - x_j|^{2\Delta_{\mathfrak{प}}}\,|x_k - x_m|^{2\Delta_{\mathfrak{प}}}} \tag{6.31}$$

From this derivation and final expression, we see the underlying reason why $g(u,v)$ in (6.21) depended only on the cross-ratios. From (3.6)–(3.7), we have that the conformal multiplet projectors $\hat{\Pi}_{\mathfrak{ब}}$ provide an expression for the basis $\{|\mathfrak{फ}_i\rangle\}$. Thus, these projectors are conformally invariant themselves (i.e., they commute with all of the conformal generators); as a result, they satisfy the conformal Ward identities in §5, and in particular satisfy the conformal Ward identities for four-point functions.

6.4 Conformal Blocks from Conformal Casimir Operators

By examining the conformal quadratic Casimir operators, we can calculate the conformal blocks directly. Here, we follow [3, 4, 41]. From §1.5, we had that the conformal algebra as isomorphic to $\mathfrak{so}(d+1,1)$ in Euclidean spacetime, with the generators $\hat{J}_{mn}$ defined by (1.11) and the algebra defined by (1.12). From this, we can construct the conformal quadratic Casimir operator given by:

$$\hat{J}^2 := -\frac{\hat{J}^{mn}\hat{J}_{mn}}{2}, \quad \hat{J}^2\,|\mathfrak{ओ}\rangle = (\Delta_{\mathfrak{ओ}}(\Delta_{\mathfrak{ओ}} - d) + s(s+d-2))\,|\mathfrak{ओ}\rangle =: \lambda_{\mathfrak{ओ}}\,|\mathfrak{ओ}\rangle \tag{6.32}$$

We note that $|\text{ओ}\rangle$ is a *generic* state in the generalised Verma module, rather than a primary or a basis vector: $\hat{J}^2$ acts with the *same* eigenvalue for *every* state (with the eigenvalue $\lambda_{\text{ओ}}$ dependent on the state's conformal dimension $\Delta_{\text{ओ}}$ and spin s). If we define the action of $\hat{J}_{mn}$ on $\hat{\text{प}}_i$ (as a differential operator) as $\mathcal{J}_{mni}$, we note that the action on $|\text{प}_i\text{प}_j\rangle = \hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle$ is given by $\mathcal{J}_{mni} + \mathcal{J}_{mnj}$:

$$\big(\mathcal{J}_{mni} + \mathcal{J}_{mnj}\big)\,|\text{प}_i\text{प}_j\rangle = \Big(\big[\hat{J}_{mn}, \hat{\text{प}}(x_i)\big]\hat{\text{प}}(x_j) + \hat{\text{प}}(x_i)\big[\hat{J}_{mn}, \hat{\text{प}}(x_i)\big]\Big)\,|0\rangle = \hat{J}_{mn}\,\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle \tag{6.33}$$

We note that the differential action of $\hat{J}_{mn}$ is given by $\hat{J}_{mn} = x_m\hat{\partial}_n - x_n\hat{\partial}_m$, due to the definition in (1.11). In terms of $\mathcal{J}_{mnj}$, we have the action of $\hat{J}^2$ on $\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle$ given by:

$$\hat{J}^2\big[\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle\big] = -\frac{(\mathcal{J}_i^{mn} + \mathcal{J}_j^{mn})(\mathcal{J}_{mni} + \mathcal{J}_{mnj})}{2}\big[\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle\big] =: \mathcal{J}_{ij}^2\big[\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,|0\rangle\big] \tag{6.34}$$

Applying this to each individual term in the expansion in (6.28), and noting that since $\hat{J}^2$ gives the same eigenvalue for every state, it intrinsically gives the same eigenvalue when acting on the left or the right, we have:

$$\begin{aligned}
\mathcal{J}_{ij}^2\,\langle\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\rangle &= \mathcal{J}_{ij}^2\,\big\langle 0\big|\mathcal{R}\{\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\}\,\hat{\Pi}_{\text{ब}}\,\mathcal{R}\{\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\}\big|0\big\rangle \\
&= \big\langle 0\big|\mathcal{R}\{\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\}\,\hat{\Pi}_{\text{ब}}\,\hat{C}\,\big[\mathcal{R}\{\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\}\big]\big|0\big\rangle \\
&= \big\langle 0\big|\mathcal{R}\{\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\}\,\hat{\Pi}_{\text{ब}}\,\hat{C}\,\mathcal{R}\{\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\}\big|0\big\rangle \\
&= \lambda_{\text{ब}}\,\big\langle 0\big|\mathcal{R}\{\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\}\,\hat{\Pi}_{\text{ब}}\,\mathcal{R}\{\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\}\big|0\big\rangle
\end{aligned} \tag{6.35}$$

(As before, $\lambda_{\text{ब}} = (\Delta_{\text{ब}}(\Delta_{\text{ब}} - d) + s_{\text{ब}}(s_{\text{ब}} + d - 2))$.) Applying this to (6.26) and (6.31), we have $C_{\text{ब}}(u, v)$ given by the eigenvalue equation:

$$\begin{aligned}
&-2\Big((1-v)^2 - u(1+v)\Big)\frac{\partial}{\partial v}\Big[v\frac{\partial C_{\text{ब}}}{\partial v}\Big] - 2\,(1-u+v)\,u\frac{\partial}{\partial u}\Big[u\frac{\partial C_{\text{ब}}}{\partial u}\Big] + 4\,(1+u-v)\,uv\frac{\partial^2 C_{\text{ब}}}{\partial u\partial v} \\
&+ 2ud\frac{\partial C_{\text{ब}}}{\partial u} - (\Delta_i - \Delta_j + \Delta_k - \Delta_m)\Big((1+u-v)\Big(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\Big) - (1-u-v)\frac{\partial}{\partial v}\Big)C_{\text{ब}} \\
&- \frac{(\Delta_i - \Delta_j)(\Delta_k - \Delta_m)(1+u-v)\,C_{\text{ब}}}{2} = -(\Delta_{\text{ब}}(\Delta_{\text{ब}} - d) + s_{\text{ब}}(s_{\text{ब}} + d - 2))\,C_{\text{ब}}
\end{aligned} \tag{6.36}$$

This gives a complicated but resolvable differential equation for $C_{\text{ब}}$, which can be solved with the appropriate boundary condition. This is given by applying $a = 0$ to (6.19), giving the boundary condition:

$$C_{\text{ब},\,\Delta_{\text{ब}},\,s=0} = \sqrt{u^{\Delta}}\left(\mathbb{1} + \frac{1}{2}x^\mu\partial_\mu + \frac{\Delta_\ell}{8(\Delta_\ell+1)}x^\mu x^\nu\partial_\mu\partial_\nu - \frac{\Delta_\ell}{8(2\Delta_\ell - d - 2)(\Delta_\ell+1)}x^2\partial^2 + \cdots\right) \tag{6.37}$$

Using the boundary condition (6.37), the differential equation (6.36) is solved in terms of hypergeometric functions [41][12]. Although these can be solved analytically in even dimensions, these have not yet been solved in closed form for odd dimensions [3, 4]. Nevertheless, in odd dimension, this equation can be exploited to perform a truncated series expansion, or alternately via recursion relations.

[12] I'm not reproducing the result here, only because of the length of the expressions. Even the variable transformation introduced in [41] to condense this expression would be overly lengthy.

6.5 Radial Quantisation of Conformal Blocks

As always, we can learn more about what's happening by radial quantisation. We put the theory in the coordinate system given in Figure 13L, with the corresponding radial quantisation given by Figure 13R; both figures are taken from [3, 4], with the coordinates given in [3, 4, 42, 43]. On this coordinate system, we see that the conformal block expansion is valid for $|\rho| < 1$. In the radial quantisation, this corresponds to putting the operators on opposing points $\pm\vec{\mathbf{n}}$ and $\pm\vec{\mathbf{m}}$ on S^{d-1}, with $\cos\theta = \vec{\mathbf{n}} \cdot \vec{\mathbf{m}}$, and with the time coordinate given as always by $\tau = -\ln r$.

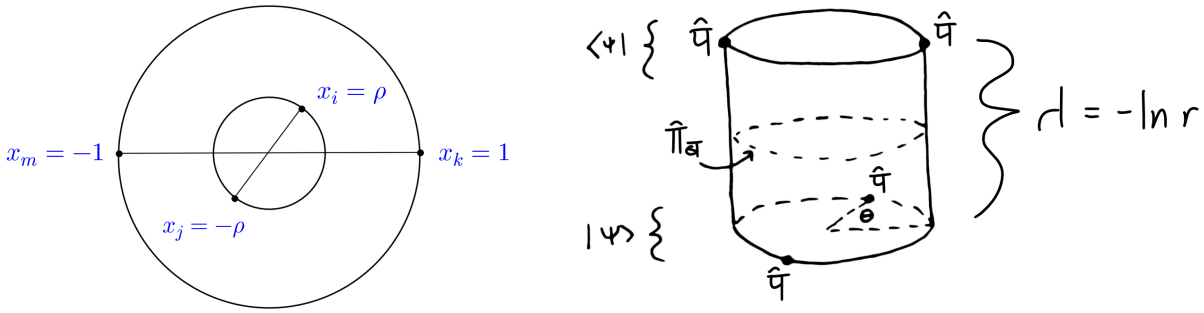

Figure 14: **Left**: Conformal transformation of coordinates used in the radial quantisation of the four-point function $\langle \hat{\Phi}(x_i)\hat{\Phi}(x_j)\hat{\Phi}(x_k)\hat{\Phi}(x_m)\rangle$.

Right: Radial quantisation of the system given by this conformal coordinate transformation. Both images modified from [3, 4].

In terms of these coordinates and the coordinate transformation given in (3.4), we define the state $|\psi\rangle$ by:

$$|\psi(\vec{\mathbf{n}})\rangle := \frac{2^{\Delta_\Phi}}{f_{\Phi\Phi\mathcal{O}}} \hat{\Phi}_{\text{cyl}}(0, \vec{\mathbf{n}})\, \hat{\Phi}_{\text{cyl}}(0, -\vec{\mathbf{n}})\, |0\rangle \tag{6.38}$$

In terms of $|\psi(\vec{\mathbf{n}})\rangle$, the conformal block is then given by:

$$C_{\mathcal{O}}(u, v) = \sum_{\mathcal{O}} \langle \psi(\vec{\mathbf{n}}) | \hat{\Pi}_{\mathcal{O}}\, \mathrm{e}^{-\tau \hat{D}} | \psi(-\vec{\mathbf{n}}) \rangle \tag{6.39}$$

The sum $\sum_{\mathcal{O}} \hat{\Pi}_{\mathcal{O}}$ came from the decomposition of the basis into a sum (3.7) over projectors onto individual conformal families indexed by the primary operators they were derived from, where in §6.2 we labelled the primaries in this sum as $\mathcal{O}$ to avoid confusion with the primaries in the four-point function. For a given primary $\mathcal{O}$ in this sum, the nth level descendant $P^{\mu_1} \cdots P^{\mu_n} |\mathcal{O}\rangle$ will have energy $\Delta_{\mathcal{O}} + n$ and spin $\jmath \in \{s+n, s+n-2, \cdots, \max(s-n, s-n \mod 2)\}$. For a descendant state $|n, \jmath\rangle^{\mu_1 \cdots \mu_\jmath}$, rotational invariance gives this state contributing to the component to $|\psi(\vec{\mathbf{n}})\rangle$ as:

$$\langle \psi(\vec{\mathbf{n}}) | n, \jmath \rangle^{\mu_1 \cdots \mu_\jmath} = c \left(\bigotimes_\ell \vec{\mathbf{n}}^{\mu_\ell} - \prod_{n,m=1}^{\jmath} \delta_{\mu_n \cdots \mu_m} \right) \tag{6.40}$$

(Here, $c \in \mathbb{C}$.) This corresponds to a rank-$\jmath$ traceless symmetric tensor. Contracting this with its Hermitian conjugate, we have $\langle \psi(\vec{\mathbf{n}}) | n, \jmath \rangle^{\mu_1 \cdots \mu_\jmath}{}_{\mu_1 \cdots \mu_s} \langle n, \jmath | \psi(\vec{\mathbf{m}}) \rangle$ corresponding to the contraction of two traceless

symmetric tensors; these contractions are given by Gegenbauer polynomials $\mathfrak{C}_\jmath^{(d-2)/2}(\vec{\mathbf{n}}\cdot\vec{\mathbf{m}})$ [3, 4]. This gives the individual contribution of the descendant state $|n,\jmath\rangle^{\mu_1\cdots\mu_\jmath}$ for $B_{n,\jmath}\in\mathbb{C}$ as:

$$\begin{aligned}\langle\psi(\vec{\mathbf{n}})|\hat{\Pi}_{\text{ब}}\,\mathrm{e}^{-\text{ड}\hat{D}}|\psi(-\vec{\mathbf{n}})\rangle &= B_{n,\jmath}\,r^{\Delta_{\text{ब}}+n}\,\langle\psi(\vec{\mathbf{n}})|n,\jmath\rangle^{\mu_1\cdots\mu_\jmath}{}_{\mu_1\cdots\mu_s}\langle n,\jmath|\psi(\vec{\mathbf{m}})\rangle\\ &= B_{n,\jmath}\,r^{\Delta_{\text{ब}}+n}\,C_\jmath^{(d-2)/2}(\cos\theta)\end{aligned}\tag{6.41}$$

Then, we have the conformal blocks given from (6.39) by the sum over $\hat{\text{ब}}$:

$$\begin{aligned}C_{\text{ब}}(u,v) &= \sum_{\text{ब}}\,\langle\psi(\vec{\mathbf{n}})|\hat{\Pi}_{\text{ब}}\,\mathrm{e}^{-\text{ड}\hat{D}}|\psi(-\vec{\mathbf{n}})\rangle\\ &= \sum_{\substack{n=0,2\cdots\\ \jmath\in\{s+n,\,s+n-2,\cdots,\max\,(s-n,\,s-n\bmod 2)\}}} B_{n,\jmath}\,r^{\Delta_{\text{ब}}+n}\,C_j^{(d-2)/2}(\cos\theta)\end{aligned}\tag{6.42}$$

The leading term in this expansion comes from $|\text{ब}\rangle$ ($n=0$, $\jmath=s$), which can be used as a boundary condition for the eigenvalue equation (6.36) to determine the remaining terms. For a unitary theory, we note that all of the coefficients $B_{n,\,s}$ must be positive, since they correspond to norms of projections of $|\psi(\vec{\mathbf{n}})\rangle$ onto energy eigenstates.

7 Symmetries and Rules for the Operator Product Expansion

7.1 Crossing Symmetry

Using the OPE, our interest is in deriving constraints from consistency conditions. As mentioned in §6.2, recursing the OPE all the way to one-point functions (6.14) gives us the important property that *all* of the properties of the CFT are given by the conformal dimensions $\{\Delta_n\}$, the spins $\{s_n\}$, and the OPE coefficients $\left\{C_{ij\ell}^{abc}\right\}$, which together defined the conformal data. However, not any random set of numbers corresponds to a valid CFT. In particular, we note that the pairing of operators in §6.3 was completely arbitrary. This is true in general: we can pair up the operators practically any way we want, and, unsurprisingly, the OPE must agree no matter what order we pick. More specifically, we require OPE associativity, described in (7.1) and depicted in Figure 16, taken from [8]:

$$\begin{gathered}\left(\left(\hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\right)\hat{\text{ओ}}_k\right) = \left(\hat{\text{ओ}}_i\left(\hat{\text{ओ}}_j\,\hat{\text{ओ}}_k\right)\right)\\ C_{ij\ell}^{abc}(x_i-x_j)\,C_{\ell mn}^{dfg}(x_j-x_m)\,\hat{\text{ओ}}_n(x_m) = C_{jm\ell}^{abc}(x_j-x_m)\,C_{i\ell n}^{dfg}(x_i-x_m)\,\hat{\text{ओ}}_n(x_m)\end{gathered}\tag{7.1}$$

In particular, if we take the correlation function of the $\hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\,\hat{\text{ओ}}_k$ product with $\hat{\text{ओ}}_m$, we can freely pair $(ij)(km)$; the consistency condition requires that this must be equivalent to the pairing $(im)(jk)$. This defines the crossing symmetry equations, depicted in Figure 15. For a system of local operators in flat space, the solutions to the crossing symmetry equations provide a complete, fully nonperturbative, description of the correlation functions without the Lagrangian[13]. We note that these conditions are the same as the second invariance condition given in (2.24).

[13] More interesting things like line and surface operators, and operators on manifolds not conformally equivalent to flat space (such as manifolds representing the system having nonzero temperature), require more conformal data.

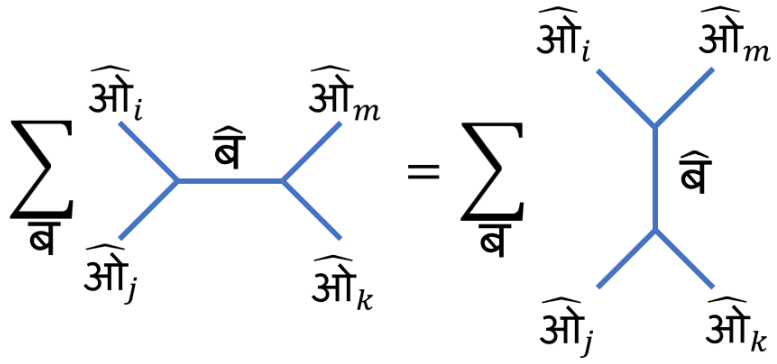

Figure 15: The OPE consistency condition requires the sum over all intermediate operators $\hat{\text{ब}}$ with the pairing $\left(\hat{\text{ओ}}_i\,\hat{\text{ओ}}_j\right)\left(\hat{\text{ओ}}_k\,\hat{\text{ओ}}_m\right)$ be equal to the sum over all intermediate operators $\hat{\text{ब}}$ with the pairing $\left(\hat{\text{ओ}}_i\,\hat{\text{ओ}}_m\right)\left(\hat{\text{ओ}}_j\,\hat{\text{ओ}}_k\right)$.

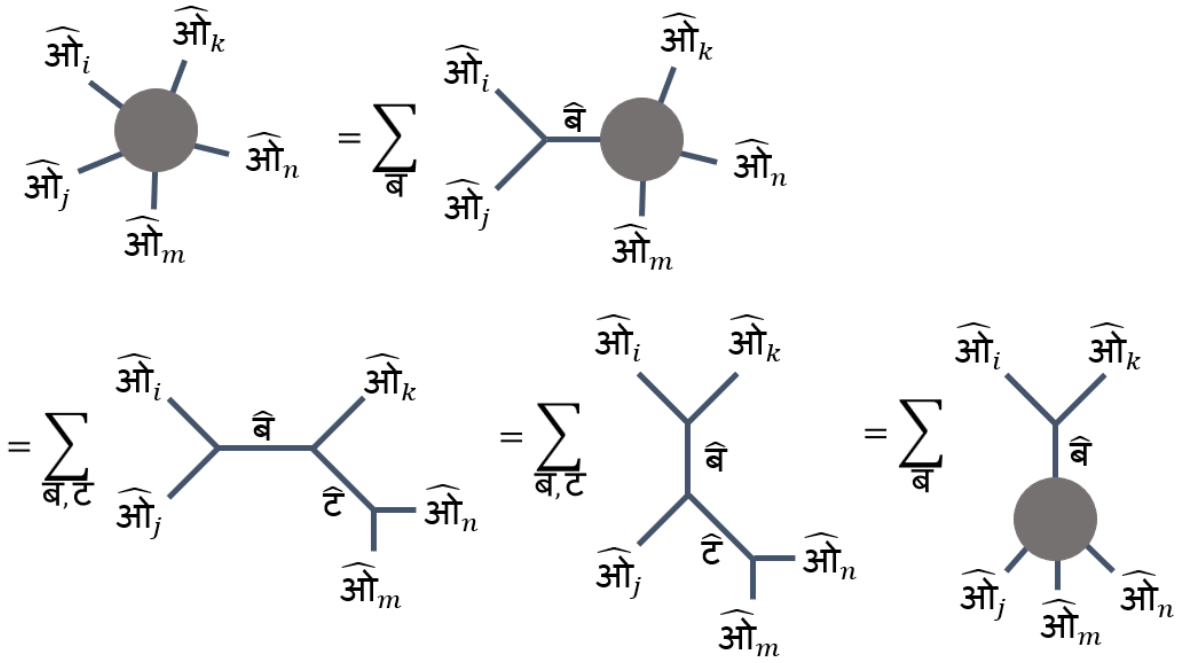

Figure 16: The OPE is associative; i.e., we can freely pair up operators associatively to recursively reduce the operator products. The OPE must be consistent under any set of pairings we choose.

As in §6, for a concrete example, we again use the case of the correlation function of identical scalar primaries $\hat{\text{प}}$ given by $\langle\hat{\text{प}}(x_i)\,\hat{\text{प}}(x_j)\,\hat{\text{प}}(x_k)\,\hat{\text{प}}(x_m)\rangle$. First, examining the radial quantisation in §6.5, applying the variable transformation $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$, and then taking the specific limiting case of $x_j \to x_i$ with all four operators colinear (given more directly by the limit $z \to 0$ with $z = \bar{z}$), we note that the blocks scale as $g_{\hat{\text{ब}}}(u,v) = cu^{\Delta_{\hat{\text{ब}}}/2} = c(z\bar{z})^{\Delta_{\hat{\text{ब}}}/2}$ (for $c \in \mathbb{C}$) in this limit.

As a result of this limit, the overall function $g(u,v)$ is dominated by $\mathbb{1}$, since this is the smallest-dimension operator in the OPE. The crossing symmetry condition corresponds to applying $u \leftrightarrow v$, or equivalently

applying $(z,\bar{z}) \mapsto (1-z, 1-\bar{z})$. Applying this, the limit $z \to 0$ gives the individual transformed conformal blocks as $z^{2\Delta_{\mathfrak{a}}} + \ln z + \cdots$, which vanishes in the $z \to 0$ limit. As a direct consequence, we see that the OPE requires an infinite number of primary operators. Along this limit, the sum is dominated by operators of dimension $\Delta = k/\sqrt{z}$ for $k \in \mathbb{C}$. As a result, $\mathbb{1}$ is mapped to the large Δ asymptotic behaviour of the sum $z^{2\Delta_{\mathfrak{a}}} + \ln z + \cdots$. Thus the crossing equation depicted in Figure 15 is satisfied as a whole, but not necessarily block-by-block: we *must* also sum over the intermediate operators that appear.

Applying the crossing symmetry depicted in Figure 15 to the four-point function expressed in terms of conformal blocks given by (6.21), we can extract a few equivalent expressions for the four-point function, which can give us extra consistency conditions for 2D CFTs. In §6.3, we paired up the operators as $(ij)(km)$, which gave us the four-point function as (6.31). If we express x_i, x_j, x_k, and x_m as points on a plane which we can parametrize by Wirtinger coordinates z and $\bar{z}$, the cross-ratios (2.22) now become $\sqrt{u}$ and $\sqrt{\bar{u}}$; i.e., the $\sqrt{v}$ cross-ratio is now expressible as the $\bar{z}$ version of the $\sqrt{u}$ ratio:

$$\sqrt{u} \coloneqq \frac{|z_1 - z_2||z_3 - z_4|}{|z_1 - z_3||z_2 - z_4|}; \quad \sqrt{v} \mapsto \sqrt{\bar{u}} \coloneqq \frac{|\bar{z}_1 - \bar{z}_2||\bar{z}_3 - \bar{z}_4|}{|\bar{z}_1 - \bar{z}_3||\bar{z}_2 - \bar{z}_4|} \tag{7.2}$$

In Wirtinger coordinates, we can factorise the four-point function into purely holomorphic and purely antiholomorphic pieces; this comes at the cost of added complexity (since we now double the number of factors in the numerator, but allows us to make simple transformations that give us added consistency conditions. Expressing (6.21) in Wirtinger coordinates, we have:

$$\left\langle \hat{\mathfrak{v}}(z_i,\bar{z}_i)\,\hat{\mathfrak{v}}(z_j,\bar{z}_j)\,\hat{\mathfrak{v}}(z_k,\bar{z}_k)\,\hat{\mathfrak{v}}(z_m,\bar{z}_m)\right\rangle = \prod_{i<j} \frac{g(u,\bar{u})}{|z_i - z_j|^{2\Delta_{\mathfrak{a}}}} \tag{7.3}$$

Performing the same pairing as (6.31) in Wirtinger coordinates (i.e., $(ij)(km)$), we have:

$$\left\langle \overbrace{\hat{\mathfrak{v}}(z_i,\bar{z}_i)\,\hat{\mathfrak{v}}(z_j,\bar{z}_j)}\,\overbrace{\hat{\mathfrak{v}}(z_k,\bar{z}_k)\,\hat{\mathfrak{v}}(z_m,\bar{z}_m)}\right\rangle = \frac{f_{ij\mathfrak{v}} f_{km\mathfrak{v}}\, C_{\mathfrak{a}}(u)\bar{C}_{\mathfrak{a}}(\bar{u})}{|z_i - z_j|^{2\Delta_{\mathfrak{a}}}\,|z_k - z_m|^{2\Delta_{\mathfrak{a}}}} \tag{7.4}$$

Meanwhile, examining the pairing $(im)(jk)$ in terms of the cross-ratios, this corresponds to $u \mapsto 1-u$ and $\bar{u} \mapsto 1-\bar{u}$:

$$\left\langle \overbrace{\hat{\mathfrak{v}}(z_i,\bar{z}_i)\,\overbrace{\hat{\mathfrak{v}}(z_j,\bar{z}_j)\,\hat{\mathfrak{v}}(z_k,\bar{z}_k)}\,\hat{\mathfrak{v}}(z_m,\bar{z}_m)}\right\rangle = \frac{f_{im\mathfrak{v}} f_{jk\mathfrak{v}}\, C_{\mathfrak{a}}(1-u)\bar{C}_{\mathfrak{a}}(1-\bar{u})}{|z_i - z_m|^{2\Delta_{\mathfrak{a}}}\,|z_j - z_k|^{2\Delta_{\mathfrak{a}}}} \tag{7.5}$$

Finally, the pairing $(ik)(km)$ corresponds to $u \mapsto 1/u$ and $\bar{u} \mapsto 1/\bar{u}$, with the overall four-point function scaled by $(u\bar{u})^{-2\Delta_{\mathfrak{a}}}$:

$$\left\langle \overbrace{\hat{\mathfrak{v}}(z_i,\bar{z}_i)\,\overbrace{\hat{\mathfrak{v}}(z_j,\bar{z}_j)\,\hat{\mathfrak{v}}(z_k,\bar{z}_k)}\,\hat{\mathfrak{v}}(z_m,\bar{z}_m)}\right\rangle = \frac{f_{ik\mathfrak{v}} f_{jm\mathfrak{v}} C_{\mathfrak{a}}\left(\frac{1}{u}\right)\bar{C}_{\mathfrak{a}}\left(\frac{1}{\bar{u}}\right)}{u^{2\Delta_{\mathfrak{a}}}\,\bar{u}^{2\Delta_{\mathfrak{a}}}\,|z_i - z_m|^{2\Delta_{\mathfrak{a}}}\,|z_j - z_k|^{2\Delta_{\mathfrak{a}}}} \tag{7.6}$$

These expressions must all be equal, giving the same set of consistency conditions given in Figure 15, but

now provided in terms of the structure constant products $\{f_{ij\text{ब}}f_{mk\text{ब}}\}$ and $\{f_{im\text{ट}}f_{jk\text{ट}}\}$. These consistency conditions are given in Figure 17.

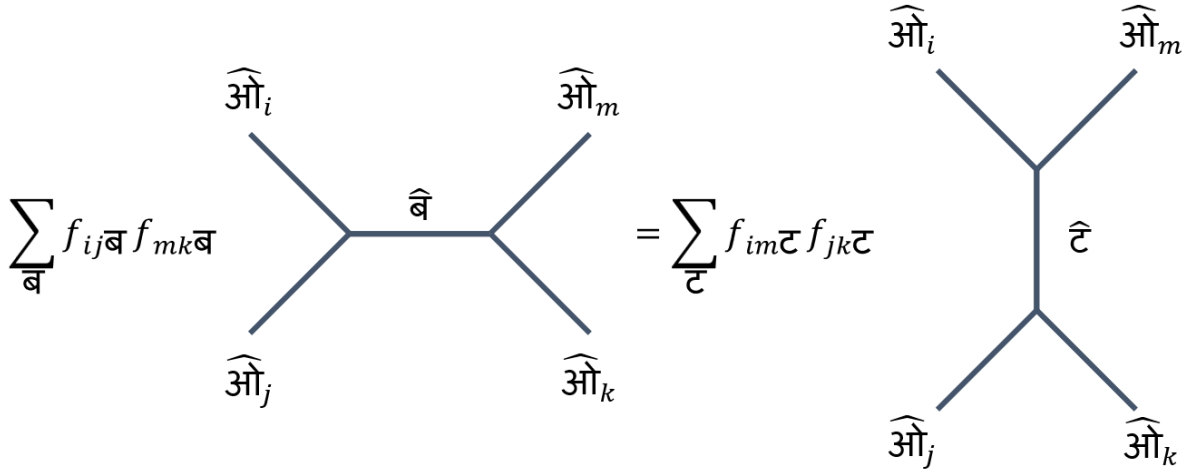

Figure 17: OPE consistency conditions for 2D CFTs, written in terms of the structure constant products $\{f_{ij\text{ब}}f_{mk\text{ब}}\}$ and $\{f_{im\text{ट}}f_{jk\text{ट}}\}$. These come from examining the symmetries of Figure 15 in 2D.

7.2 Fusion Rules for Unitary Minimal Models

In §3.3 and §4.5, we saw that operators whose conformal dimensions were at the unitarity bounds (3.33) corresponded to Verma modules that were reducible representations of the Virasoro algebra. In particular, we saw that these operators give null states whose sectors in the Verma module decoupled from the rest of the Virasoro algebra, having no descendants and corresponding to zero values of the Kač determinant (4.23), $M_N(h,c) = 0$. As mentioned there, for $c < 1$, these correspond to unitary minimal models (with RCFTs being a subclass of these), which are notable for having additional structure properties. By examining the Kač determinant for specific n-point functions, we can extract this structure, which provides a lot of information about the structure of OPEs in unitary minimal models[14].

Starting with the conformal Ward identity for scale invariance for primary operators (5.28), we consider the action of L_{-m} as differential operators on the expectation value of products of chiral spinless primaries (i.e., spinless primaries which only depend on z); we derive this relation first. Applying (5.10) to (5.28),

[14] In principle, we could have done all of the calculations in this section right after §5.4. Since the conclusions will be applied to OPEs of unitary minimal models, however, it made more sense to me to do this after introducing the concept of OPEs generally.

we have $\langle L_{-m}\,\hat{\mathfrak{q}}(\mathfrak{z})\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle$ given by:

$$\begin{aligned}\langle L_{-m}\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle &= \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}(\mathfrak{z})}\mathrm{d}z\,(z-\mathfrak{z})^{1-m}\,\langle T(z)\,\hat{\mathfrak{q}}(\mathfrak{z})\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle \\ &= -\sum_{i=1}^{n}\oint_{\mathcal{C}(\mathfrak{z}_i)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}(z-\mathfrak{z})^{1-m}\left(\frac{\Delta_i}{(z-\mathfrak{z})^2}+\frac{1}{z-\mathfrak{z}}\frac{\partial}{\partial\mathfrak{z}_i}\right)\langle\hat{\mathfrak{q}}(\mathfrak{z})\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle \qquad (7.7)\\ &= -\sum_{i=1}^{n}\left(\frac{(1-m)\Delta_i}{(\mathfrak{z}_i-z)^m}+(\mathfrak{z}_i-\mathfrak{z})^{1-m}\frac{\partial}{\partial\mathfrak{z}_i}\right)\langle\hat{\mathfrak{q}}(\mathfrak{z})\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle\end{aligned}$$

Here, $\mathcal{C}(\mathfrak{z})$ is the contour depicted in the left-hand side of Figure 18 and $\{\mathcal{C}(\mathfrak{z}_i)\}$ is the continuous deformation of $\mathcal{C}(\mathfrak{z})$ to the contour set depicted in the right-hand side of this figure, where the minus sign in the second line of (7.7) comes about from the contour deformation. Thus, we have the action of L_{-m} defined by the differential operator $\mathbb{L}_{-m}$ as:

$$\mathbb{L}_{-m}\,\langle\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle = -\sum_{i=1}^{n}\left(\frac{(1-m)\Delta_i}{(\mathfrak{z}_i-z)^m}+(\mathfrak{z}_i-\mathfrak{z})^{1-m}\frac{\partial}{\partial\mathfrak{z}_i}\right)\langle\hat{\mathfrak{q}}(\mathfrak{z})\,\hat{\mathfrak{q}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{q}}_n(\mathfrak{z}_n)\rangle \qquad (7.8)$$

From this expression, we have $\mathbb{L}_{-1}$ as simply the sum of the individual derivatives with respect to $\mathfrak{z}_i$:

$$\mathbb{L}_{-1} = -\sum_{i=1}^{n}\left(\frac{(1-1)\Delta_i}{(\mathfrak{z}_i-z)^1}+(\mathfrak{z}_i-\mathfrak{z})^{1-1}\frac{\partial}{\partial\mathfrak{z}_i}\right) = -\sum_{i=1}^{n}\frac{\partial}{\partial\mathfrak{z}_i} \qquad (7.9)$$

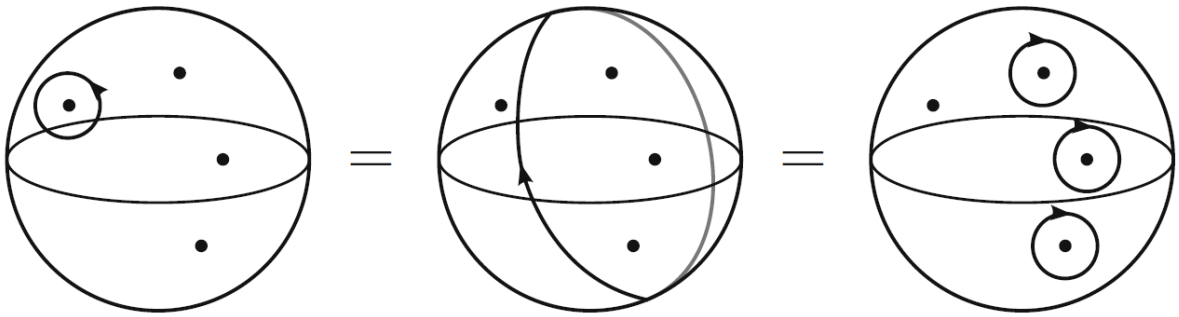

Figure 18: The contour $\mathcal{C}(\mathfrak{z})$ in (7.7) (on the left-hand side), which is continuously deformed into the contour $\mathcal{C}(\mathfrak{z})$ (on the right-hand side) in this derivation. The change in contour directionality adds a minus sign. Taken from [1].

We note that (7.8) isn't exactly the same as (4.3), since (4.3) is L_{-m} as a generator of the algebra, whereas this is the action of L_{-m} on chiral spinless primaries. This is analogous to how the action of Poincaré generators appears slightly differently as a generator and as an operator on fields of various spin values. In particular, we see that (7.8) is the same as (4.3) when $\Delta_i = 0$ in (7.8). This is unsurprising, since Δ_i is the is the eigenvalue of the dilation operator, so it appears from (4.3) when considering L_0.

We can use (7.8) to re-express (4.28) as a differential equation, which will allow us to see the effect of the level 2 null state in the operator product of chiral spinless primary fields. From (4.28), we have the presence of the level 2 in a general operator product of chiral spinless primary fields given by:

$$\begin{gathered}\left(\mathbb{L}_{-2}-\frac{3}{2(2\Delta+1)}\mathbb{L}_{-1}\right)\left\langle\hat{\mathfrak{v}}(\mathfrak{z})\,\hat{\mathfrak{v}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{v}}_n(\mathfrak{z}_n)\right\rangle=0\\ \sum_{i=1}^{n}\left(\frac{\Delta_{\mathfrak{v}}}{(\mathfrak{z}_i-\mathfrak{z})^2}-\frac{1}{\mathfrak{z}_i-\mathfrak{z}}\frac{\partial}{\partial\mathfrak{z}_i}-\frac{3}{2(2\Delta_{\mathfrak{v}}+1)}\frac{\partial^2}{\partial\mathfrak{z}_i^2}\right)\left\langle\hat{\mathfrak{v}}_1(\mathfrak{z}_1)\cdots\hat{\mathfrak{v}}_n(\mathfrak{z}_n)\right\rangle=0\end{gathered}\tag{7.10}$$

Applying this to the two-point function (2.12) gives us no meaningful information:

$$\begin{gathered}\left(\mathbb{L}_{-2}-\frac{3}{2(2\Delta+1)}\mathbb{L}_{-1}\right)\left\langle\hat{\mathfrak{v}}(\mathfrak{z})\,\hat{\mathfrak{v}}_1(\mathfrak{z}_1)\right\rangle=0\\ \left(\frac{\Delta_{\mathfrak{v}}}{(\mathfrak{z}_1-\mathfrak{z})^2}-\frac{1}{\mathfrak{z}_i-\mathfrak{z}}\frac{\partial}{\partial\mathfrak{z}_1}-\frac{3}{2(2\Delta_{\mathfrak{v}}+1)}\frac{\partial^2}{\partial\mathfrak{z}_1^2}\right)\frac{k}{|\mathfrak{z}_1-\mathfrak{z}|^{2\Delta_{\mathfrak{v}}}}=0\\ \left(\Delta_{\mathfrak{v}}+2\Delta_{\mathfrak{v}}-\frac{3}{2(2\Delta_{\mathfrak{v}}+1)}2\Delta_{\mathfrak{v}}(2\Delta_{\mathfrak{v}}+1)\right)\frac{k}{|\mathfrak{z}_1-\mathfrak{z}|^{2\Delta_{\mathfrak{v}}}}=(\Delta_{\mathfrak{v}}+2\Delta_{\mathfrak{v}}-3\Delta_{\mathfrak{v}})\frac{\hbar}{|\mathfrak{z}_1-\mathfrak{z}|^{2\Delta_{\mathfrak{v}}}}=0\end{gathered}\tag{7.11}$$

For the three-point function (2.16), on the other hand, we can apply (7.10) to (2.16). We leave the full calculation for the unabridged version, and simply report the results; this gives a constraints on the conformal dimensions $\Delta\coloneqq\Delta_{\mathfrak{v}}$, Δ_1, and Δ_2, which we can write as a constraint for Δ_2 in terms of Δ and Δ_1:

$$\begin{gathered}2(2\Delta+1)(\Delta+2\Delta_2-\Delta_1)=3(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)\\ \Delta_2=\frac{1}{6}+\frac{\Delta}{3}+\Delta_1\pm\frac{2}{3}\sqrt{\Delta^2+3\Delta\Delta_1-\frac{\Delta}{2}+\frac{3\Delta_1}{2}+\frac{1}{16}}\end{gathered}\tag{7.12}$$

We can apply these constraints to unitary minimal models by considering the general properties of Kač determinant at level 2. From (4.22), we have the Kač matrix at level 2 as:

$$M_2(\Delta_{\mathfrak{v}},c)=\begin{pmatrix}\langle\mathfrak{v}|L_2L_{-2}|\mathfrak{v}\rangle & \langle\mathfrak{v}|L_1L_1L_{-2}|\mathfrak{v}\rangle\\ \langle\mathfrak{v}|L_2L_{-1}L_{-1}|\mathfrak{v}\rangle & \langle\mathfrak{v}|L_1L_1L_{-1}L_{-1}|\mathfrak{v}\rangle\end{pmatrix}\tag{7.13}$$

The individual elements are:

$$\langle\mathfrak{v}|L_2L_{-2}|\mathfrak{v}\rangle=\left\langle\mathfrak{v}\middle|\frac{c}{2}+4L_0\middle|\mathfrak{v}\right\rangle=\frac{c}{2}+4\Delta_{\mathfrak{v}}$$

$$\langle\mathfrak{v}|L_1L_1L_{-2}|\mathfrak{v}\rangle=\langle\mathfrak{v}|(L_1)(3L_{-1})|\mathfrak{v}\rangle=6\Delta_{\mathfrak{v}};\quad\langle\mathfrak{v}|L_2L_{-1}L_{-1}|\mathfrak{v}\rangle=\langle\mathfrak{v}|(3L_1)(L_{-1})|\mathfrak{v}\rangle=6\Delta_{\mathfrak{v}};$$

$$\begin{aligned}\langle\mathfrak{v}|L_1L_1L_{-1}L_{-1}|\mathfrak{v}\rangle&=\langle\mathfrak{v}|L_1[L_1,L_{-1}]L_1|\mathfrak{v}\rangle+\langle\mathfrak{v}|L_1L_{-1}L_1L_{-1}|\mathfrak{v}\rangle\\&=\langle\mathfrak{v}|(L_1)(2L_0)(L_{-1})|\mathfrak{v}\rangle+\langle\mathfrak{v}|[L_1,L_{-1}][L_1,L_{-1}]|\mathfrak{v}\rangle\\&=2\langle\mathfrak{v}|L_1[L_0,L_{-1}]|\mathfrak{v}\rangle+4{\Delta_{\mathfrak{v}}}^2+4{\Delta_{\mathfrak{v}}}^2=4\Delta_{\mathfrak{v}}+8{\Delta_{\mathfrak{v}}}^2\end{aligned}\tag{7.14}$$

The Kač determinant at level 2 is then:

$$\begin{aligned}\det M_2(\Delta_{\mathfrak{q}}, c) &= 32\Delta_{\mathfrak{q}}\left({\Delta_{\mathfrak{q}}}^2 - \frac{5\Delta_{\mathfrak{q}}}{8} + \frac{\Delta_{\mathfrak{q}} c}{8} + \frac{c}{16}\right) \\ &= 32\Delta_{\mathfrak{q}}\left(\Delta_{\mathfrak{q}} - \frac{5 - c + \sqrt{(1-c)(25-c)})}{16}\right)\left(\Delta_{\mathfrak{q}} - \frac{5 - c - \sqrt{(1-c)(25-c)})}{16}\right)\end{aligned} \tag{7.15}$$

Following (4.26), we label these solutions by $\Delta_{r,s}$; this labelling will help us understand how these solutions fit into the broader concept of minimal models:

$$\Delta_{1,1} = 0, \quad \Delta_{1,2} = \frac{5 - c - \sqrt{(1-c)(25-c)}}{16}, \quad \Delta_{2,1} = \frac{5 - c + \sqrt{(1-c)(25-c)}}{16} \tag{7.16}$$

From (7.12), we already had a relationship between $\Delta := \Delta_{\mathfrak{q}}$, Δ_1, and Δ_2. If we select $\Delta_{\mathfrak{q}} = \Delta_{2,1}$ and $\Delta_1 = \Delta_{p,q}$, then we see that the two possibilities for Δ_2 have the form $\Delta_{p-1,q}$ and $\Delta_{p+1,q}$. Thus, at *most* two of the $\{f_{ijk}\} = \{f_{\mathfrak{q}\mathfrak{q}_1\mathfrak{q}_2}\}$ in (2.16) will be nonzero. This, remarkably, restricts the OPEs of conformal *families*. If $[\hat{\mathfrak{q}}]$ denotes the conformal family of $\hat{\mathfrak{q}}$, then the OPE of fields in $[\hat{\mathfrak{q}}_2] = [\hat{\mathfrak{q}}_{(2,1)}]$ with fields in any other $[\hat{\mathfrak{q}}_{(p,q)}]$ is given *solely* by fields in $[\hat{\mathfrak{q}}_{(p+1,q)}]$ and fields in $[\hat{\mathfrak{q}}_{(p-1,q)}]$:

$$[\hat{\mathfrak{q}}_{(2,1)}][\hat{\mathfrak{q}}_{(p,q)}] = \sum_i c_i [\hat{\mathfrak{q}}_{(p+1,q),i}] + \sum_j c_j [\hat{\mathfrak{q}}_{(p+1,q),j}] \tag{7.17}$$

(As before, the square brackets denote that this is true for *any* field within the conformal family defined by the corresponding primary. The product of the operators on the left-hand side are implicitly taken to be inside a correlator.) This is a fusion rule for unitary minimal models, and is generally written as:

$$[\hat{\mathfrak{q}}_{(2,1)}] \times [\hat{\mathfrak{q}}_{(p,q)}] = [\hat{\mathfrak{q}}_{(p+1,q)}] + [\hat{\mathfrak{q}}_{(p+1,q)}] \tag{7.18}$$

Unitary minimal models, as mentioned before, have a finite number of primaries. By examining higher-level Kač determinants until we exhaust the theory, we retrieve the full fusion rules, which form a closed algebra:

$$[\hat{\mathfrak{q}}_{(p_1,q_1)}] \times [\hat{\mathfrak{q}}_{(p_2,q_2)}] = \sum_{\substack{k=1+|p_1-p_2| \\ k+p_1+p_2 \text{ odd}}}^{p_1+p_2-1} \sum_{\substack{\ell=1+|q_1-q_2| \\ \ell+q_1+q_2 \text{ odd}}}^{q_1+q_2-1} [\hat{\mathfrak{q}}_{(k,\ell)}] \tag{7.19}$$

For arbitrary RCFTs, we can generalise the fusion rules to the OPE of arbitrary conformal families. RCFTs follow a fusion algebra, given for arbitrary conformal families $[\hat{\mathfrak{q}}_i]$ and $[\hat{\mathfrak{q}}_j]$ by:

$$[\hat{\mathfrak{q}}_i] \times [\hat{\mathfrak{q}}_j] = \sum_k N_{ij}^k [\hat{\mathfrak{q}}_k] \tag{7.20}$$

Here, $N_{ij}^k \in \mathbb{N}$, with $N_{ij}^k = 0$ only if $\{f_{ijk}\} = 0$. This algebra is associative and commutative. From the commutativity of the algebra, we have $N_{ij}^k = N_{ji}^k$; meanwhile, from the associativity of the algebra, we have $[\hat{\mathfrak{q}}_i] \times ([\hat{\mathfrak{q}}_j] \times [\hat{\mathfrak{q}}_k]) = ([\hat{\mathfrak{q}}_i] \times [\hat{\mathfrak{q}}_j]) \times [\hat{\mathfrak{q}}_k]$. Applying the fusion algebra to both sides of this

expression gives the fusion sum rule:

$$\begin{gathered}
[\hat{\mathfrak{q}}_i] \times ([\hat{\mathfrak{q}}_j] \times [\hat{\mathfrak{q}}_k]) = [\hat{\mathfrak{q}}_i] \times \left(\sum_\ell N_{jk}^\ell [\hat{\mathfrak{q}}_\ell] \right) = ([\hat{\mathfrak{q}}_i] \times [\hat{\mathfrak{q}}_j]) \times [\hat{\mathfrak{q}}_k] = \left(\sum_\ell N_{ij}^\ell [\hat{\mathfrak{q}}_\ell] \right) \times [\hat{\mathfrak{q}}_k] \\
\sum_\ell N_{jk}^\ell [\hat{\mathfrak{q}}_i] \times [\hat{\mathfrak{q}}_\ell] = \sum_{\ell,m} N_{jk}^\ell N_{i\ell}^m [\hat{\mathfrak{q}}_m] = \sum_\ell N_{ij}^\ell [\hat{\mathfrak{q}}_\ell] \times [\hat{\mathfrak{q}}_k] = \sum_{\ell,m} N_{ij}^\ell N_{\ell k}^m [\hat{\mathfrak{q}}_m] \qquad (7.21) \\
\sum_{ell} N_{kj}^\ell N_{i\ell}^m = \sum_\ell N_{ij}^\ell N_{ik}^m
\end{gathered}$$

Here, the last line is the fusion sum rule, which we derived since the second-to-last line holds in general across any set of $[\hat{\mathfrak{q}}_m]$.

7.3 Fusion and Braiding with Conformal Blocks

In §7.1, we derived the crossing symmetry equations both for OPEs generally and for 2D CFTs. In the latter, by writing n-point functions in Wirtinger coordinates, we were able to factorise them into holomorphic and antiholomorphic pieces, at the cost of doubling the number of factors in the expressions. For RCFTs, the symmetry equations give us extra structure, which are incredibly useful for practical calculations. Since once again the original reference puts it best, I'll quote [1] directly:

> For RCFTs a simplification occurs, as there are only a finite number of conformal families which can propagate as intermediate states. This means that the conformal blocks for the three different channels form a finite-dimensional vector space. The crossing symmetry then says that the different classes of conformal blocks are nothing else than three different choices of basis which must be related by linear transformations.

The equivalence of the $(ij)(km)$ pairing (7.4) and the $(im)(jk)$ pairing (7.5) corresponds to a conformal block change of basis known as the fusion matrix F, defined as:

$$C_{\mathfrak{q}}(u) = \sum_{\mathfrak{r}} F_{ijkm}^{\mathfrak{pr}} C_{\mathfrak{r}}(1-u) \qquad (7.22)$$

We can understand this expression better graphically; this is given in Figure 19. Similarly, the equivalence of the $(ij)(km)$ pairing (7.4) and the $(ik)(jm)$ pairing (7.6) corresponds to a conformal block change of basis known as the braiding matrix B, defined as:

$$C_{\mathfrak{q}}(u) = \sum_{\mathfrak{r}} B_{ijkm}^{\mathfrak{pr}} C_{\mathfrak{r}}\left(\frac{1}{u}\right) \qquad (7.23)$$

As before, we can understand this expression better graphically; this is given in Figure 20.

The symmetries of the five-point function under fusion and braiding give important identities. First, we have the commutativity of the diagrams given in Figure 21 (taken from [44]). This gives rise to the pentagon identity. If we keep the operator indices $ijkl$ from left to right at the top of Figure 21 and the

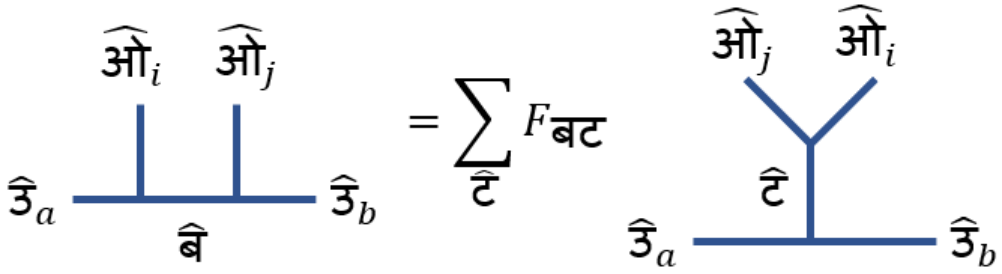

Figure 19: Fusion matrix for RCFTs, corresponding to a change of conformal block basis given by (7.22).

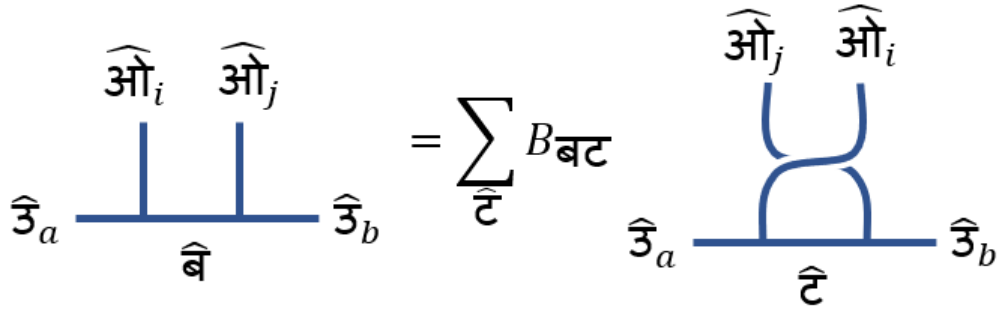

Figure 20: Braiding matrix for RCFTs, corresponding to a change of conformal block basis given by (7.23).

index n at the bottom, we have the commutativity of Figure 21 giving the pentagon identity as:

$$F^{ac}_{ijkb} F^{bd}_{cmin} = \sum_e F^{be}_{kman} F^{ad}_{jein} F^{ec}_{jkdm} \tag{7.24}$$

Meanwhile, we have the commutativity of the diagrams given in Figure 22 (modified from [44]). This gives rise to the hexagon identity. Keeping the same labelling as before, we have the commutativity of Figure 22 combined with the pentagon identity (7.24) giving the hexagon identity as:

$$\sum_e B^{ae}_{ijkb} B^{bc}_{jmen} B^{ed}_{kmic} = \sum_f B^{bf}_{kman} B^{ad}_{jmif} B^{fc}_{jkdn}. \tag{7.25}$$

Quite surprisingly, these relations appear in $(2+1)$-D topological theories, as identities for excitations of Chern-Simons models..

8 Numerical Techniques for the Conformal Bootstrap

8.1 Conformal Bootstrap

Here, we briefly discuss the numerical techniques for the conformal bootstrap; as before, we primarily follow [3, 4]. As mentioned in §6.3, the decomposition of $g(u, v)$ into conformal blocks given by (6.25) is analogous to the expansion of a problem in E&M or QM in the natural basis of $\mathcal{L}^2$. This allows us to reformulate the crossing equation depicted in Figure 15 (or, equivalently, the second condition in

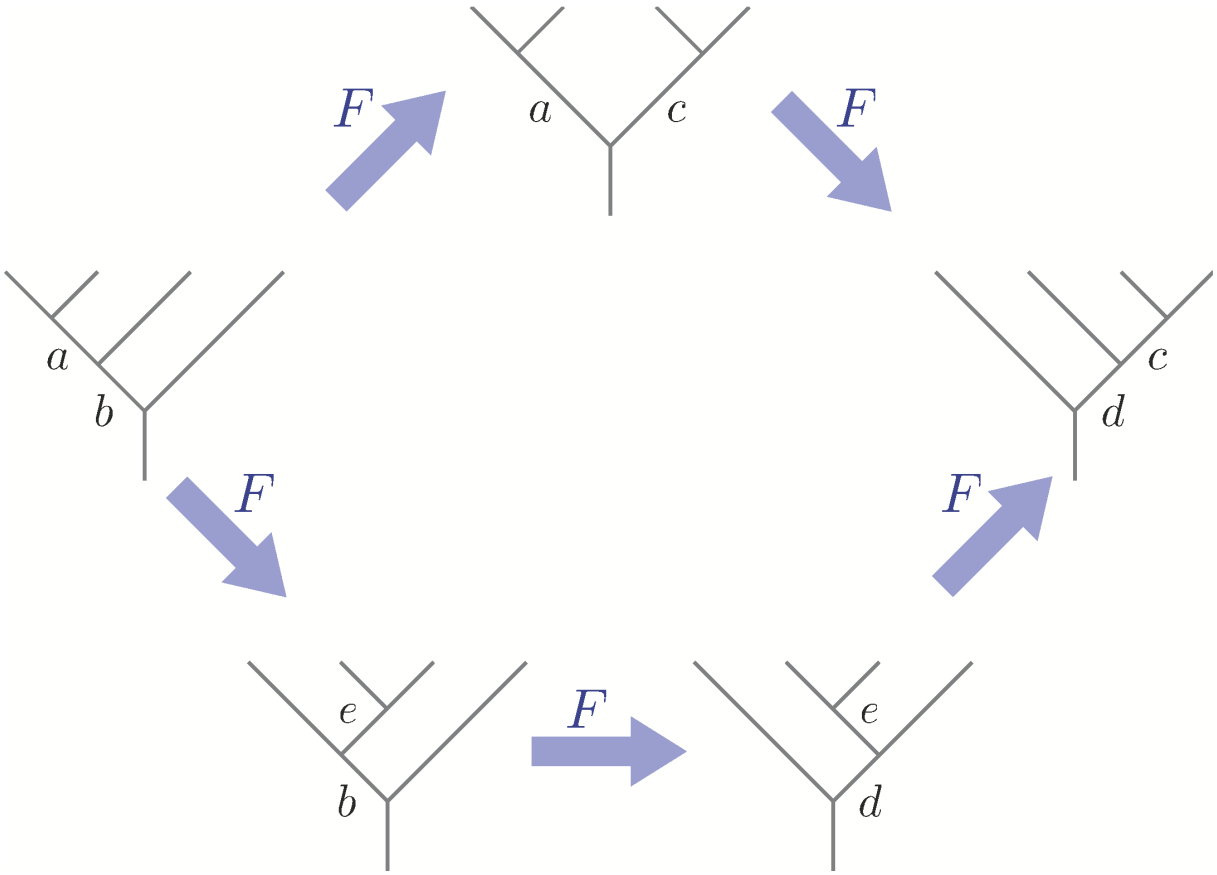

Figure 21: Pentagon identity for the fusion matrix. Here, we implicitly keep the labels $ijkl$ from left to right at the top and the index n at the bottom. Taken from [44].

(2.24)) as a linear-algebraic expression rather than trying to solve it *exactly* [45]. From this, we can derive bounds on the CFT data, and thus allows us to make statements on some of the data (in particular, the smallest operator dimension values) without having to calculate the entire thing. Starting again with the four-point function $\langle \hat{\mathfrak{v}}(x_i)\,\hat{\mathfrak{v}}(x_j)\,\hat{\mathfrak{v}}(x_k)\,\hat{\mathfrak{v}}(x_m)\rangle$ of identical scalar primaries $\hat{\mathfrak{v}}$, we can rewrite the crossing equation as:

$$\sum_{\mathfrak{w}} f^2_{\mathfrak{v}\mathfrak{v}\mathfrak{w}} \underbrace{\left(v^{\Delta_{\mathfrak{v}}} C_{\mathfrak{w}}(u,v) - u^{\Delta_{\mathfrak{v}}} C_{\mathfrak{w}}(v,u)\right)}_{F^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s}(u,v)} := \sum_{\mathfrak{w}} f^2_{\mathfrak{v}\mathfrak{v}\mathfrak{w}} F^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}(u,v) = 0 \tag{8.1}$$

Defining the functions $F^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}(u,v)$ in this fashion allows us to interpret the crossing equation as an equation for the linear dependence of vectors. Specifically, since the $f^2_{\mathfrak{v}\mathfrak{v}\mathfrak{w}}$ are all positive, we can interpret $F^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}(u,v)$ as vectors $\vec{\mathbf{F}}^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}$ in the infinite-dimensional vector space of possible functions $f(u,v)$ of u and v, with the crossing equation corresponding to the equation that determines whether or not the $\left\{\vec{\mathbf{F}}^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}\right\}$s are a linearly dependent set:

$$\sum_{\Delta_{\mathfrak{w}},\, s_{\mathfrak{w}}} f^2_{\mathfrak{v}\mathfrak{v}\mathfrak{w}} \vec{\mathbf{F}}^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}} = 0 \tag{8.2}$$

This may or may not be possible, depending on what the $\left\{\vec{\mathbf{F}}^{\Delta_{\mathfrak{v}}}_{\Delta_{\mathfrak{w}} s_{\mathfrak{w}}}\right\}$s are. If they're linearly dependent, then this equation is satisfied and the conformal data we're giving *does* correspond to a valid CFT; if

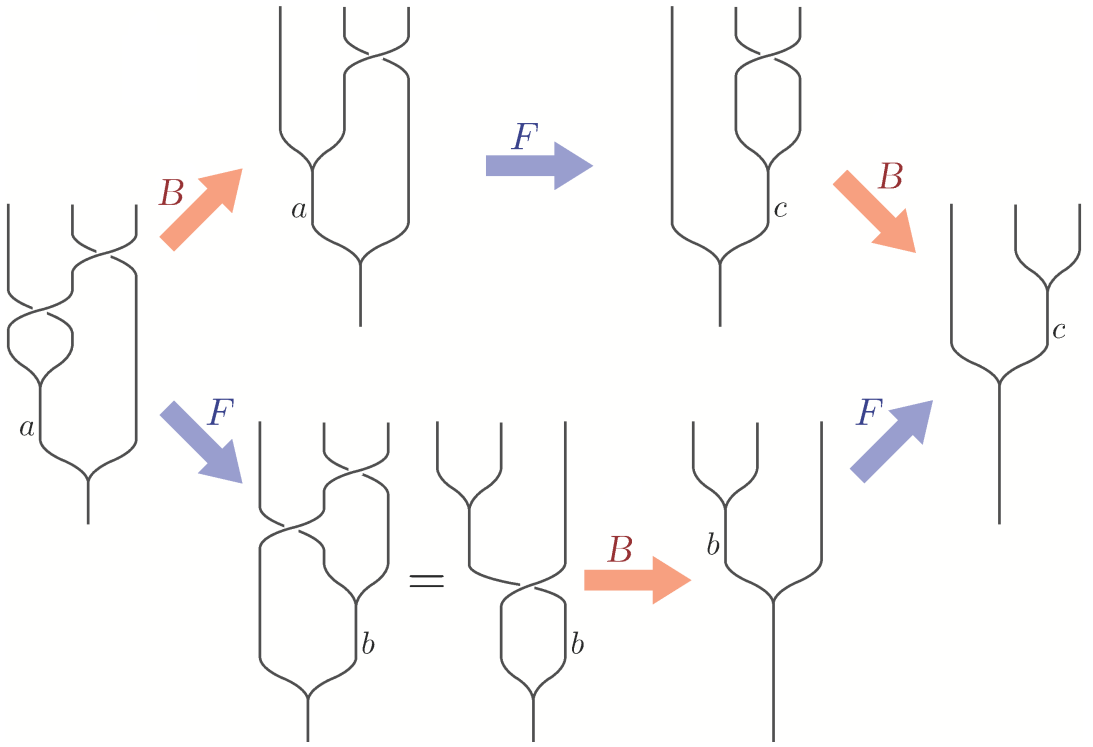

Figure 22: Hexagon identity for the fusion and braiding matrices. Here, we use the same labelling as in Figure 21. Modified from [44] (the original labels the braiding matrices as "R").

they're not linearly dependent, then it's not. The statement of linear dependence is equivalent to stating that there exists a separating plane α such that all of the $\left\{\vec{\mathbf{F}}^{\Delta_{\text{प}}}_{\Delta_{\text{ब}} s_{\text{ब}}}\right\}$s lie on one side of the plane; this is depicted in Figure 23. As a result, if a separating plane exists, then the proposed conformal data does not give a valid CFT.

Recasting the crossing equation in this way directly gives rise to an algorithm, which we can use to implement this condition numerically:

1. Start with a proposed set of conformal data (i.e., a proposed set of conformal dimensions $\{\Delta_n\}$, the spins $\{s_n\}$, and the OPE coefficients $\left\{C_{ij\ell}^{abc}\right\}$).

2. Try to find a linear nonnegative functional α that acts on all of the $\left\{\vec{\mathbf{F}}^{\Delta_{\text{प}}}_{\Delta_{\text{ब}} s_{\text{ब}}}\right\}$ such that for all of them, we have $\alpha\left[\vec{\mathbf{F}}^{\Delta_{\text{प}}}_{\Delta_{\text{ब}} s_{\text{ब}}}\right] \geq 0$.

3. If α exists, then the conformal data does not give a valid CFT.

We can use the algorithm to bound the OPE coefficients as well [3, 4, 46], by examining the *separating cone* Λ; the algorithm stays the same, and we have the linear dependency relation and functional relation for a specific operator $\hat{\text{ओ}}$ given respectively by:

$$1 - f^2_{\text{पपओ}} F^{\Delta_{\text{प}}}_{d\Delta_{\text{ओ}} s_{\text{ओ}}}(u,v) = \sum_{\text{ब}} f^2_{\text{पपब}} F^{\Delta_{\text{प}}}_{\Delta_{\text{ब}} s_{\text{ब}}}(u,v) = 0, \quad \Lambda\left[F^{\Delta_{\text{प}}}_{d\Delta_{\text{ओ}} s_{\text{ओ}}}\right] \geq 0, \quad \Lambda[1] = 1 \tag{8.3}$$

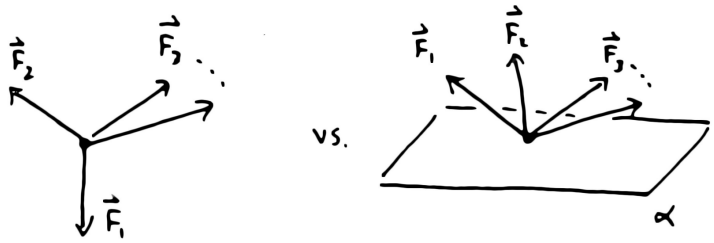

Figure 23: The linear algebraic interpretation (8.2) of the crossing equation corresponds to determining whether a separating plane α exists such that all of the $\left\{\vec{\mathbf{F}}^{\Delta_{\mathfrak{q}}}_{d\Delta_{\mathfrak{a}}s_{\mathfrak{a}}}\right\}$s lie on one side of the plane. If such a plane exists, then the proposed conformal data does not give rise to a valid CFT. Taken from [3, 4].

8.2 Conformal Bootstrap Implementation

These algorithms have the obvious problem that the second step (finding α or Λ, respectively) is incredibly difficult in general: both the space that the $\left\{\vec{\mathbf{F}}^{\Delta_{\mathfrak{q}}}_{\Delta_{\mathfrak{a}}s_{\mathfrak{a}}}\right\}$s live on, and the space of functionals that α and Λ live on are infinite-dimensional. Specifically, we have an infinite number of constraints (corresponding to each distinct possible $\Delta_{\mathfrak{a}}$ and $s_{\mathfrak{a}}$, the first being uncountably infinite and the second being countably infinite), as well as the space of *all* possible functionals. As is often the case (especially numerically), this is remedied by restricting ourselves to finite subspaces of the infinite spaces we're actually dealing with; this trades off rigour in our statements for tractability. In particular, if we can't find the functionals, we can no longer *definitely* say that this corresponds to a valid set of conformal data: either it *is* a valid set of conformal data, or our artificial restrictions ended up excluding the values that would have given the seperating plane / cone.

Each of these infinities is typically handled in different ways [3, 4, 9]:

- For the continuous functional space, restrict α (or Λ) to be determined by a linear combination of derivatives of $F(z,\bar{z})$ around the crossing-symmetric point $z=\bar{z}=1/2$, up to a finite derivative order p:

$$\alpha(F) = \sum_{m+n\leq p} \alpha_{mn} \frac{\partial^m}{\partial z^m}\frac{\partial^n}{\partial \bar{z}^n} F(z,\bar{z})\Big|_{z=\bar{z}=1/2} \tag{8.4}$$

  Thus, α is now parametrised by a *finite* number of coefficients α_{mn}, which can be searched over computationally.

- For the continuous infinity of the conformal dimensions $\Delta_{\mathfrak{a}}$:

  – Discretise the set of $\Delta_{\mathfrak{a}}$s being searched and impose an upper cutoff. This turning the search over $\{\Delta_{\mathfrak{a}}\}$s into a finite set of linear equations, which can be minimised using matrix techniques for simultaneous linear equations; given by linear optimisation.

  – Directly construct an n-simplex (i.e., n-hypertriangle) from a finite set of $\left\{\vec{\mathbf{F}}^{\Delta_{\mathfrak{q}}}_{\Delta_{\mathfrak{a}}s_{\mathfrak{a}}}\right\}$s, and use the continuous simplex algorithm (which uses Newton's method to find an optimal simplex over a continuous interval of $\{\Delta_{\mathfrak{a}}\}$s and discrete set of spins).

  – In addition to the finite derivative order approximation for α, approximate the $\alpha\left[\vec{\mathbf{F}}^{\Delta_{\mathfrak{q}}}_{\Delta_{\mathfrak{a}}s_{\mathfrak{a}}}\right] \geq 0$ con-

straints as a set of polynomial inequalities for all $y \geq 0$, and use semidefinite programming:

$$\alpha(F) = \sum_{m+n \leq p} \alpha_{mn} P_s^{mn}(\Delta_s^{\min} + y) \geq 0 \tag{8.5}$$

- For the countable infinity of spins $s_{\mathfrak{q}}$, impose a finite maximum spin.

To close out this excursion, we provide two specific examples of the bootstrap in action. For a 2D CFT with a real-valued scalar primary $\mathfrak{q}$ with conformal dimension $\Delta_{\mathfrak{q}} = 1/8$, we define the vectors $\vec{\mathbf{v}}$ and conformal blocks $C_{\mathfrak{q}}$ by:

$$\begin{aligned}
\vec{\mathbf{v}}(F) &= \Big(H(1/2, 3/5) - H(1/2, 1/3), \quad H(1/2, 3/5) - H(1/3, 1/4)\Big) \in \mathbb{R} \\
H(z, \bar{z}) &:= \left.\frac{F(u, v)}{u^{\Delta_{\mathfrak{q}}} - v^{\Delta_{\mathfrak{q}}}}\right|_{\Delta_{\mathfrak{q}} = 1/8}, \quad u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \\
C_{\mathfrak{q}} &= k_+(z)\, k_-(\bar{z}) + k_-(z) k_+(\bar{z}), \quad k_\pm := x^{(\Delta_{\mathfrak{q}} \pm s_{\mathfrak{q}})/2}\, {}_2F_1\Big(\frac{\Delta_{\mathfrak{q}} \pm s_{\mathfrak{q}}}{2}, \frac{\Delta_{\mathfrak{q}} \pm s_{\mathfrak{q}}}{2}, \Delta_{\mathfrak{q}} \pm s_{\mathfrak{q}}, x\Big)
\end{aligned} \tag{8.6}$$

As discussed in §8.1, the vectors $\vec{\mathbf{v}}(F)$ defined this way sum to zero (i.e., are linearly dependent) only if we have a valid CFT with that corresponding conformal data. The plots of the vanishing $\{\vec{\mathbf{v}}(F)\}$ are given in Figure 24, with the stress-energy tensor identified amongst the possible operators, as well as the separating plane and the solutions on a different side of the plane as the rest of the possible theories.

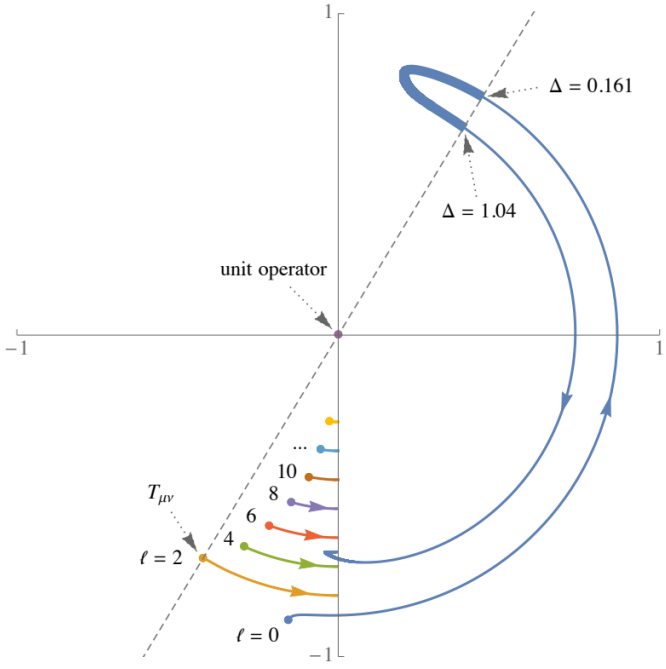

Figure 24: Plots of vanishing $\{\vec{\mathbf{v}}(F)\}$, corresponding to valid sets of conformal data. Larger dots correspond to vectors located *at* the unitarity bound $\Delta_{\mathfrak{q}} = s_{\mathfrak{q}}$, with the stress-energy tensor $\mathfrak{q} = T_{\mu\nu}$ identified as a primary operator. The thicker region of the curve corresponds to a different half-plane (identified by the dashed line) from the remaining operators. Taken from [3, 4].

More generally, we can use the bootstrap technique to search for valid CFTs across smallest scaling dimension, and combine this technique with other properties (such as specific symmetries) to further restrict the space of theories being searched. As an example, restricting ourselves to $\mathbb{Z}_2$ theories with odd primary $\mathfrak{q}$, we can pick a value of the minimum scaling dimension Δ_0 such that all of the scalars in the $\mathfrak{q} \times \mathfrak{q}$ OPE have dimension $\Delta \geq \Delta_0$. Then, applying the finite derivative power technique (8.4), we can identify theories based on the power of $\Delta_\mathfrak{q}$ as a function of Δ_0; these are shown in Figure 25. Notably, we see that the very physically valuable example of the 2D Ising model corresponds to a kink in the allowable region boundary as we increase the derivative power; this interest in kinks is a general feature of the numerical bootstrap approach [3, 4].

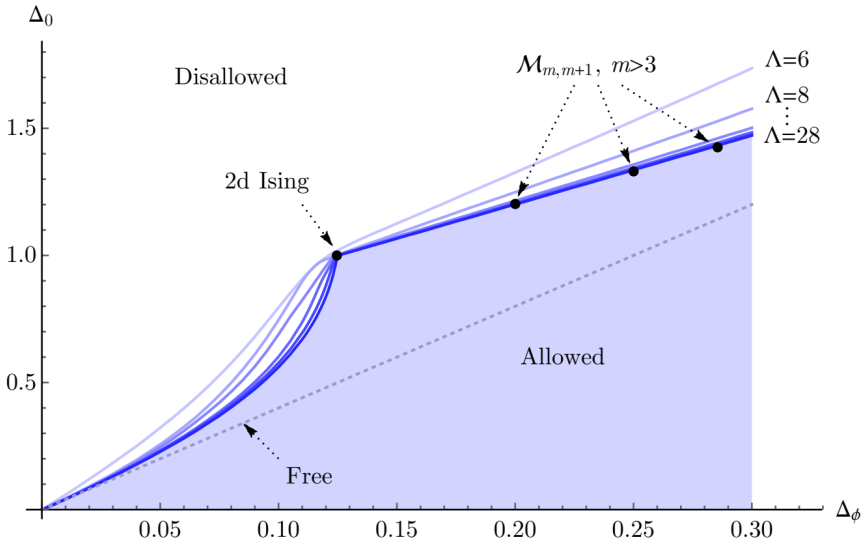

Figure 25: Upper bounds on the dimension Δ_0 of the smallest dimension scalar primary in the $\mathfrak{q} \times \mathfrak{q}$ OPE as a function of $\Delta_\mathfrak{q}$ (labelled here as Δ_ϕ). Λ here denotes the power p of the derivative in (8.4); several different values are provided, with the light blue region corresponding to the allowed theories underneath the largest power ($p = 28$). The dashed line denotes the lowest-dimension scalar primary in the free boson model, whose scalar primary operators are given by $\cos(k\mathfrak{q})$. The 2D Ising model, as well as certain unitary minimal models, have been identified. Taken from [3, 4].

9 Chern-Simons Theory and CFTs

This section discusses Chern-Simons theory, a type of topological quantum field theory that arises as the effective theory of strongly correlated electrons in fractional quantum Hall states. We might ask why this seemingly-unrelated theory is being discussed in a discussion of CFT; however, we'll see that *all* of the properties of Chern-Simons theory come from CFT, and in particular CFT calculations will allow us to solve any problem we have involving Chern-Simons.

9.1 Chern-Simons and the Winding Number

A topological quantum field theory is a physical theory described by a Hamiltonian and/or a Lagrangian density in which all quantities of the system depend only on topological invariants. The statement that every quantity depends solely on topological invariants is equivalent to saying that all observables must

be independent of any local quantities. Thus, nothing can depend on changes of the metric tensor $g_{\mu\nu}$, since $g_{\mu\nu}$ defines the local properties of the space that the theory lives on.

We can make this statement concrete by considering the stress-energy tensor $T^{\mu\nu}$, defined as the functional derivative of the action S with respect to the metric tensor:

$$T^{\mu\nu} := \frac{\delta S}{\delta g_{\mu\nu}} \tag{9.1}$$

Then, saying that every physical quantity must be independent of local quantities, and must depend only on topological invariants, is equivalent to saying that the variation of S with respect to $g_{\mu\nu}$ is necessarily zero. In other words, we *always* have $T^{\mu\nu} = 0$. This might be concerning at first: if every component of the stress-energy tensor is always zero, then the energy – the T^{00} component – must also always be zero, and thus the Hamiltonian (the quantum mechanical energy operator) must always be zero as well. However, we know that a Lagrangian constructed solely out of n-forms will also be invariant under changes of $g_{\mu\nu}$ [47, 48].

If we try to build n-forms out of a generic gauge field A_μ, we can perhaps hope that we can construct a Lagrangian that has some nonzero physics. For a Lagrangian to represent valid physics, at least one term must have spacetime derivatives (to represent kinetic energy), and this term must yield nonzero physics by itself. Thus, the simplest possible n-form we can start with is the exterior derivative $\mathrm{d}A$; i.e., a 2-form. For $n = 2$, we have either $\mathrm{d}A = \varepsilon^{\mu\nu}\partial_\mu A_\nu$ or $A \wedge A = \varepsilon^{\mu\nu} A_\mu A_\nu$. For $\mathrm{d}A$, Stokes' theorem gives us $\int_{\mathcal{M}} \mathrm{d}A = \int_{\partial\mathcal{M}} A$ over any manifold $\mathcal{M}$; thus, all of the information of the theory lives on the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$. For manifolds without boundary, A is necessarily zero. Even for manifolds with boundary, however, this term gives zero: since A_μ is a gauge field, we can always find a gauge transformation that makes A_μ vanish no matter what A_μ we write down.

For $n = 3$, on the other hand, the two possibilities are $A \wedge \mathrm{d}A = \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ or $A \wedge A \wedge A = \varepsilon^{\mu\nu\rho} A_\mu A_\nu A_\rho$. In isolation, the first term gives rise to nonzero physics; combining it with the second term gives a different (but closely related) nonzero physical model generalizing the physics of the first term. The first term is the Abelian Chern-Simons (ACS) theory; $\mathcal{L}_{\text{ACS}} = \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. Combining the first and second terms, meanwhile, gives us the non-Abelian Chern-Simons (NACS) theory [49–53] over the manifold $\mathcal{M}$, given by the Lagrangian:

$$S_{\text{NACS}} = \int_{\mathcal{M}} \mathrm{d}^d x \, \mathcal{L}_{\text{NACS}} = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}^d x \, \varepsilon^{\mu\nu\rho} \operatorname{Tr}\left[A_\mu \partial_\nu A_\rho - \frac{2\mathrm{i}}{3} A_\mu A_\nu A_\rho \right] \tag{9.2}$$

(Here, $A_\mu := A_\mu^a \mathfrak{t}^a$ is defined over the Lie group G, with corresponding Lie algebra $\mathfrak{g}$. $\mathfrak{t}^a \in \mathfrak{g}$ are the generators of G with commutation relation $[\mathfrak{t}^a, \mathfrak{t}^b] = \mathrm{i} f^{abc} \mathfrak{t}^c$, and the trace in S_{NACS} is in the algebra $\mathfrak{g}$.) In the rest of this section, we will focus on the NACS Lagrangian specifically (and drop the "NA"), due to its generality: everything that we derive for NACS (and the properties that relate NACS to conformal field theories and holography) will *also* be true for ACS.

Before continuing, we note that we constructed the Chern-Simons Lagrangians out of 3-forms. As a result, this Lagrangian can only be defined on odd-dimensional manifolds. Physically, this means we require an

even number of dimensions plus one time dimension. Although Chern-Simons theory is of some interest to the physics of fundamental particles and strings, we can also recognize this as an effective theory for two-dimensional systems; i.e., a Lagrangian that emerges from strong electron-electron correlations in condensed matter systems where electron motion is confined by a strong confining potential to two spatial dimensions. In fact, Chern-Simons theories are precisely the effective theories that emerge from the collective fermion behavior of electrons in fractional quantum Hall states [52, 54–56]. The detailed physics of the fractional quantum Hall states will be skipped here, since it's not relevant to examining the properties of Chern-Simons models[15].

As always, we can construct the path integral to determine the physical quantities. Applying the NACS Lagrangian (9.2) to the definition of the path integral given by $Z = \int \mathcal{D}A_\mu \exp\{\mathrm{i}S/\hbar\}$, this gives:

$$Z_{\mathrm{CS}} = \int \mathcal{D}A_\mu \exp\left\{\frac{\mathrm{i}S_{\mathrm{CS}}}{\hbar}\right\} = \int \mathcal{D}A_\mu \exp\left\{\frac{-\mathrm{i}k}{4\pi\hbar}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\Big[A_\mu\partial_\nu A_\rho - \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\rho\Big]\right\} \tag{9.3}$$

For NACS to be a valid physical theory, we require that the path integral remain invariant under gauge transformations; in other words, under the gauge transformation, the path integral must acquire no more than an overall phase $\mathrm{e}^{2n\pi\mathrm{i}} = 1$ (with $n \in \mathbb{Z}$) to Z. For $g \in G$, the gauge transformation for A_μ is given by $A_\mu \mapsto g^{-1}A_\mu g + \mathrm{i}g^{-1}\partial_\mu g$. Under this transformation, the Lagrangian transforms as:

$$S_{\mathrm{CS}} \mapsto S_{\mathrm{CS}} + \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\Big(\partial_\mu \mathrm{Tr}\big[(\partial_\nu g)g^{-1}A_\rho\big] + \frac{1}{3}\mathrm{Tr}\big[g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\big]\Big) \tag{9.4}$$

For the path integral to remain invariant under this transformation, the extra terms in the action must contribute an overall phase to the path integral:

$$Z = \int \mathcal{D}A_\mu \exp\left\{\frac{\mathrm{i}S_{\mathrm{CS}}}{\hbar}\right\} \longmapsto$$

$$\int \mathcal{D}A_\mu \exp\left\{\frac{\mathrm{i}S_{\mathrm{CS}}}{\hbar} + \frac{\mathrm{i}k}{4\pi\hbar}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\,\partial_\mu \mathrm{Tr}\big[(\partial_\nu g)g^{-1}A_\rho\big] + \frac{\mathrm{i}k}{12\pi\hbar}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\big]\right\}$$

$$\longmapsto \mathrm{e}^{2n\pi\mathrm{i}}\,\mathrm{e}^{2m\pi\mathrm{i}}\int \mathcal{D}A_\mu \exp\left\{\frac{\mathrm{i}S_{\mathrm{CS}}}{\hbar}\right\}$$

We thus *require* both terms to be integers, in order for this transformation to contribute nothing more than an overall phase to the path integral. Focusing first on the second term in (9.4) induced by the gauge transformation, we can define this term as proportional to the winding number:

$$w(g) = \frac{1}{24\pi^2\hbar}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\big] \tag{9.5}$$

If we have $G = \mathrm{SU}(N)$, the winding number must be an integerThis then forces $k \in \mathbb{Z}$. We note that *any*

[15] In general, this is the advantage of effective field theories: we can write down Lagrangians that completely capture the physics at the scale we're looking at, without having to worry about the microscopic dynamics.

$w(g) \in \mathbb{Z}$ and $k \in \mathbb{Z}$ will leave the path integral invariant. In terms of $w(g)$ the overall transformations to S_{CS} for any $w(g), k \in \mathbb{Z}$ is given by:

$$\begin{aligned} S_{\text{CS}} &\mapsto S_{\text{CS}} + \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}^d x \left(\partial_\mu \mathrm{Tr}\left[\varepsilon^{\mu\nu\rho} (\partial_\nu g) g^{-1} A_\rho \right] + 2\pi\hbar k w(g) \right) + \frac{k}{4\pi} \cdot \frac{1}{3} \cdot 24\hbar\pi^2 \\ &= S_{\text{CS}} + \mathfrak{Q}_{\partial\mathcal{M}} + 2\pi\hbar k w(g) \end{aligned} \tag{9.6}$$

(In the first line, we also pulled $\varepsilon^{\mu\nu\rho}$ into the overall derivative for the first term in (9.4), and wrote it as $\mathfrak{Q}(\partial\mathcal{M})$, in preparation for looking at that term shortly.) The corresponding transformation to Z is then:

$$Z = \int \mathcal{D}A_\mu \exp\left\{ \frac{\mathrm{i}S_{\text{CS}}}{\hbar} \right\} \mapsto \int \mathcal{D}A_\mu \exp\left\{ \frac{\mathrm{i}S_{\text{CS}}}{\hbar} + \frac{2\pi\mathrm{i}\hbar k w(g)}{\hbar} \right\} = \mathrm{e}^{2\pi\mathrm{i}k w(g)} \int \mathcal{D}A_\mu \exp\left\{ \frac{\mathrm{i}S_{\text{CS}}}{\hbar} \right\} \tag{9.7}$$

The winding number provides a concrete sense of what is meant by a "large" gauge transformation: a gauge transformation that gives $w(g) \in \mathbb{Z} \setminus 0$ will keep the path integral the same, but *not* the action itself. Large gauge transformations wind around the spacetime manifold $\mathcal{M}$; this is depicted in Figure 26.

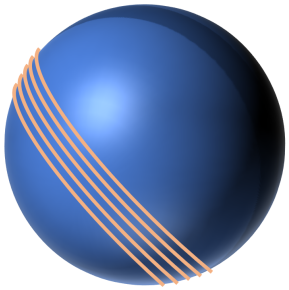

Figure 26: A large gauge transformation (orange) can wind around a compact manifold (blue) multiple times. The winding number (9.5) counts the number of times this transformation wraps around the manifold.

Ordinarily, (9.7) also gives us a condition on k: since we know $w(g) \in \mathbb{Z}$, we're required to have $k \in \mathbb{Z}$ in order to keep the states supporting the Chern-Simons theory from failing. (This is the origin of fractionalized quantum Hall conductance.) Conversely, we can interpret (9.7) as a condition on gauge transformations: for *fixed* $k, w(g) \in \mathbb{Z}$, anything that contributes a large gauge transformation (in the sense of changing the value of $w(g)$ by an integer) can keep the Chern-Simons states from failing but can mess up the internal dynamics of the theory.

9.2 Chern-Simons and Wess-Zumino-Novikov-Witten Theories

Turning our attention now to the first term in (9.4), which we wrote in (9.6) as $\mathfrak{Q}_{\partial\mathcal{M}}$, we saw there that we could pull $\varepsilon^{\mu\nu\rho}$ inside the derivative; i.e., the entirety of $\mathfrak{Q}_{\partial\mathcal{M}}$ is a derivative term. Again by Stokes' theorem, we have $\int_{\mathcal{M}} \partial_\mu f = \int_{\partial\mathcal{M}} f$; i.e., this entire term lives on the boundary of $\mathcal{M}$. In quantum field theory, we would ordinarily set the boundary to zero, making this vanish identically; however, in a condensed matter system, we can no longer do this. Instead, we can cycle through the terms in the trace

(pulling $\varepsilon^{\mu\nu\rho}$ back out of the trace) and integrate by parts to get:

$$\begin{aligned}\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^d x\, \partial_\mu \mathrm{Tr}\left[\varepsilon^{\mu\nu\rho}(\partial_\nu g)g^{-1}A_\rho\right] &= \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{(d-1)}x\, \mathrm{Tr}\left[\varepsilon^{\mu\nu\rho}(\partial_\nu g)g^{-1}A_\rho\right] \\ &= \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{(d-1)}x\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[g^{-1}A_\rho(\partial_\nu g)\right] \\ &= \frac{k\lambda}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{(d-1)}x\, \mathrm{Tr}\left[g^{-1}(\partial_\mu g)g^{-1}(\partial^\mu g)\right]\end{aligned} \tag{9.8}$$

(Here, $\lambda \in \mathbb{R}$.)

It might look as though we've swapped one inscrutable expression for another in doing this. However, this term is a very famous model, the nonlinear sigma model (NLSM) [57], which exhibits *classical* conformal symmetry. The emergence of a field theory with classical conformal symmetry at the boundary term of Chern-Simons strongly suggests that states at the edge of Chern-Simons theories are conformal field theories (CFTs). The fact that this was derived from the *first* term in (9.4); i.e., the kinetic term, which we need to be nonzero in the Lagrangian to have a sensible theory, suggests that this is a general property of Chern-Simons theories (both Abelian and non-Abelian). These suspicions are enhanced when we put the final form of the boundary variation in (9.8) back into the variation in (9.4):

$$\delta S_{\mathsf{CS}} = \frac{k\lambda}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{(d-1)}x\, \mathrm{Tr}\left[g^{-1}(\partial_\mu g)g^{-1}(\partial^\mu g)\right] + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{d}^d x\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\right] \tag{9.9}$$

This is, itself, the action for a specific class of CFTs called the Wess-Zumino-Novikov-Witten (WZNW) theories. Explicitly, this tells us that the boundaries of NACS theories are WZNW CFTs, whose behaviour (conserved charges, currents, etc.) depend on the winding number. (This is, in fact, true for abelian Chern-Simons as well.) k is, specifically, the WZNW central charge, which tells us the specific kind of WZNW model we're dealing with.

By itself, the derivation of (9.9) from (9.4) is already quite powerful: it tells us that the properties of the WZNW CFT help determine essential properties of NACS, and vice-versa. In particular, we see that every Chern-Simons model has a WZNW CFT associated with it on its edges, and that large NACS gauge transformations in the bulk ($\mathcal{M}$) can change the values of WZNW currents on the boundary ($\partial\mathcal{M}$). The relation between the bulk theory and the boundary theory is even stronger, however, which we can see by examining the Chern-Simons wavefunctions in $d = 2+1$ (i.e., the Chern-Simons wavefunctions in two spatial and one time dimension).

Writing (9.2) in $d = 2+1$ explicitly in terms of its individual components, writing $\mathcal{M} = \Sigma \times \mathbb{R}$ (where Σ is our 2D planar surface with boundary $\partial\Sigma$ and $\mathbb{R}$ is time), and fixing the gauge as $A_0 = \varphi_{\mathsf{CS}} = 0$, we

have:

$$S_{\text{CS}} = \frac{k}{4\pi} \int\limits_{\Sigma \times \mathbb{R}} \mathrm{d}^3x \, \mathrm{Tr}\left[A_0(\partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i) - A_i \frac{\mathrm{d}A_j}{\mathrm{d}t} + A_j \frac{\mathrm{d}A_i}{\mathrm{d}t} + \frac{2\mathrm{i}}{3} A_0 A_i A_j \right]$$
$$S_{\text{CS},\, A_0 = 0} = \frac{k}{4\pi} \int\limits_{\Sigma \times \mathbb{R}} \mathrm{d}^3x \, \mathrm{Tr}\left[-A_i \frac{\mathrm{d}A_j}{\mathrm{d}t} + A_j \frac{\mathrm{d}A_i}{\mathrm{d}t} \right] = -\frac{k}{4\pi} \int\limits_{\Sigma \times \mathbb{R}} \mathrm{d}^3x \, \varepsilon^{ij} \, \mathrm{Tr}\left[A_i \frac{\mathrm{d}A_j}{\mathrm{d}t} \right] \tag{9.10}$$

We note that the final expression comes from the kinetic term in (9.2), so this is the $(2+1)$-d Lagrangian for both the abelian and non-Abelian Chern-Simons theories. From the final expression, we can directly write down the Poisson bracket and canonical commutation relation (CCR):

$$\{A_i(\vec{\mathbf{x}}), A_j(\vec{\mathbf{y}})\} = \varepsilon_{ij} \, \delta^{ab} \, \delta^2(\vec{\mathbf{x}} - \vec{\mathbf{y}}), \qquad [A_i(\vec{\mathbf{x}}), A_j(\vec{\mathbf{y}})] = \frac{2\pi\mathrm{i}}{k} \varepsilon_{ij} \, \delta^{ab} \, \delta^2(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \tag{9.11}$$

(Here, δ^{ab} is over the algebra indices in $\mathfrak{g}$.) A notable curiosity is that the components of A_μ are canonically conjugate to each other.

We could, in principle, try to construct the Chern-Simons wavefunctions from the Hamiltonian. However, from (9.10) and the fact that the components of A_μ are canonically conjugate to each other, we see that the Lagrangian is already in the form $L = \pi \dot{\mathfrak{o}}$ (where π is the canonical conjugate momentum of $\mathfrak{o}$ and $\dot{\mathfrak{o}}$ is the time derivative of $\mathfrak{o}$). This gives the Hamiltonian $H = \pi\dot{\mathfrak{o}} - L = 0$, as we expect from requiring $T^{\mu\nu} = 0$ earlier.

Luckily, we can also construct the wavefunctions from the CCR, which gives us a crucial result. Without loss of generality, we can select A_i as the "coordinate" and A_j as the "conjugate momentum". Thus, A_j acts on wavefunctions in "position" space as functional derivatives with respect to A_i:

$$A_j \, |\psi(A_i)\rangle = -\frac{2\pi\mathrm{i}}{k} \frac{\delta \, |\psi\rangle}{\delta A_i} \tag{9.12}$$

Next, we can consider the field strength tensor F_{ij}, which in terms of (9.12) explicitly has the form:

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = -\frac{2\pi\mathrm{i}}{k} \left(\frac{\partial}{\partial x^i} \frac{\delta}{\delta A_i} - \frac{\partial}{\partial x^j} A_i \right) - \frac{4\pi^2}{k^2} \left(A_i \frac{\delta}{\delta A_i} - \frac{\delta}{\delta A_i} A_i \right) \tag{9.13}$$

By definition, F_{ij} must generate unitary gauge transformations U on Σ:

$$U = \exp\left\{ \mathrm{i} \int \mathrm{d}^3x \, \theta^{ij} F_{ij} \right\} \tag{9.14}$$

(Here, θ^{ij} is the tensor of infinitesimal parameters corresponding to the gauge transformation strength.) Since U is a gauge transformation, on *physical* states, we require $U = \mathbb{1}$:

$$U \, |\psi(A_i)\rangle = \exp\left\{ \mathrm{i} \int \mathrm{d}^3x \, \theta^{ij} F_{ij} \right\} |\psi(A_i)\rangle = \mathbb{1} \, |\psi(A_i)\rangle = |\psi(A_i)\rangle \tag{9.15}$$

Now, taking the derivative ∂_k of (9.15) will give us back F_{ij}, and gives us that the action of F_{ij} on physical

states $|\psi(A_i)\rangle$ is zero; which we can combine with (9.13) to get a functional differential equation:

$$\begin{gathered}
\frac{\partial[U\,|\psi(A_i)\rangle]}{\partial x^k} = \frac{\partial U}{\partial x^k}\,|\psi(A_i)\rangle + U\frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} = \frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} \\
\frac{\partial}{\partial x^k}\left[\exp\left\{\mathrm{i}\int \mathrm{d}^3x\,\theta^{ij}F_{ij}\right\}\right]|\psi(A_i)\rangle + \mathbb{1}\frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} = \frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} \\
\theta^{ij}F_{ij}\,|\psi(A_i)\rangle + \frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} = \frac{\partial\,|\psi(A_i)\rangle}{\partial x^k} \qquad (9.16) \\
F_{ij}\,|\psi(A_i)\rangle = -\frac{2\pi\mathrm{i}}{k}\left(\frac{\partial}{\partial x^i}\frac{\delta\,|\psi(A_i)\rangle}{\delta A_i} - \frac{\partial[A_i\,|\psi(A_i)\rangle]}{\partial x^j}\right) - \frac{4\pi^2}{k^2}\left(A_i\frac{\delta\,|\psi(A_i)\rangle}{\delta A_i} - \frac{\delta\,|\psi(A_i)\rangle}{\delta A_i}A_i\right) = 0 \\
\left(\frac{\partial}{\partial x^i}\frac{\delta}{\delta A_i} + \frac{\partial}{\partial x^i}A_i - \frac{\mathrm{i}k}{2\pi}\left[A_i, \frac{\partial}{\partial A_i}\right]\right)
\end{gathered}$$

(In writing the second-to-last line, $F_{ij}\,|\psi(A_i)\rangle = 0$, we used the fact that θ^{ij} are parameters; i.e., not dependent on the "coordinate" A_i.)

This is a functional differential equation that we can solve to get $|\psi(A_i)\rangle$, the Chern-Simons wavefunctions. We can either solve this the hard way (directly) or a much easier way, by using the connection to the WZNW CFT from earlier and seeing if we can gain any insight from that. Remarkably, this differential equation is *directly* solved by the WZNW path integral, meaning that the Chern-Simons wavefunctions are, identically, the WZNW path integrals:

$$\begin{aligned}
|\psi_{\mathrm{CS}}(A_i)\rangle = \int \mathcal{D}g \exp\Bigg\{ & \frac{k\lambda}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^{(d-1)}x\,\mathrm{Tr}\big[g^{-1}(\partial_\mu g)g^{-1}(\partial^\mu g)\big] \\
& + \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{d}^d x\,\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\big[g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\big] + k\int_{\mathcal{M}} \mathrm{d}^d x\,\mathrm{Tr}\big[A_\mu(\partial^\mu g)g^{-1}\big]\Bigg\} \qquad (9.17)
\end{aligned}$$

The importance of this result is difficult to overstate, both for topological quantum computation and for physics in general. All of the properties of the system can be determined by examining the time evolution of the system's wavefunction spectrum, or by examining the path integral of the system. (9.17) explicitly gives us a correspondence between the Chern-Simons wavefunctions and the WZNW path integrals; thus, computing any quantity in a Chern-Simons theory automatically computes a specific quantity in a corresponding WZNW theory, and vice versa, and properties of the Chern-Simons theory determine the corresponding WZNW theory, and vice-versa. This is a much stronger relationship than (9.9), and one of the strongest statements in physics; this is the bulk-boundary CS-WZNW correspondence. As a result, when trying to calculate the properties of CS theories, we can instead calculate properties of WZNW theories or vice-versa.

9.3 Anyons in 2D

As mentioned in §9.2, the CS-WZNW correspondence (9.17) tells us that the properties of the Chern-Simons theory and spectrum are uniquely specified by the corresponding WNZW theory. The particle

excitations of $(2+1)$-d Chern-Simons theory are anyons, whose properties are entirely determined by the boundary WNZW theory. These are a class of particle unique to $(2+1)$ dimensions, which have a different type of exchange statistics than bosons and fermions. In particular, whereas bosons acquire a positive sign upon two successive interchanges and fermions acquire a minus sign, anyon exchange involves a more general phase factor $e^{i\theta}$ for any $\theta \in \mathbb{R}$ (hence the name anyon).

We can understand this heuristically first. In a system with identical particles, we can define an exchange operator $\hat{E}$ as an operator that swaps two indistinguishable particles: $\hat{E}\,|1,2\rangle = |2,1\rangle$. Then, applying $\hat{E}$ twice gives us the same state: $\hat{E}^2\,|1,2\rangle = |1,2\rangle$. Thus, the eigenvalues of $\hat{E}^2$ are $e^{i\theta}$ for any real angle $\theta \in \mathbb{R}$. We can think of a single application of $\hat{E}$ as equivalent to a half-circle rotation followed by a translation of both:

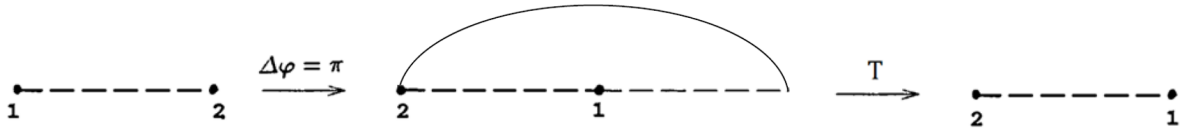

Figure 27: For indistinguishable particles in any dimension, a single exchange operation corresponds to a rotation of one particle around the other, followed by a translation of both. Modified from [50].

In this framework, $\hat{E}^2$ corresponds to rotating particle 2 completely around particle 1 (or vice versa), with the path of particle 2 creating a loop around particle 1:

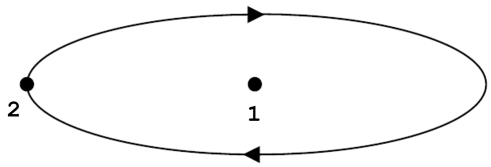

Figure 28: In light of Figure 27, two successive exchange operations correspond to looping one particle around the other.

This loop defines a plane, with particle 1 coplanar with the loop. In three (or more) dimensions, we can continuously deform the loop out of the plane, and contract it down to a single point. Conversely, in two dimensions, we can't, and particle 1 serves as a fundamental obstruction to this contraction:

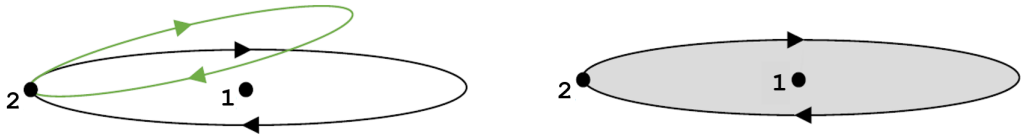

Figure 29: Circling particle 2 around particle 1 creates a loop that defines a plane, with particle 1 coplanar to the plane. In $d \geq (3+1)$, we can continuously deform this loop out of the plane and contract it to a point, which is not possible in $d = (2+1)$.

Quite surprisingly, this argument is not only heuristic, but indeed a genuine property of the path integral. For a general system (i.e., a system with an as-of-yet-unspecified Lagrangian $\mathcal{L}$ in arbitrary dimension),

we can consider the position-space path integral where we specify the endpoints as equal to the starting points, such that ${x_f}^\mu = {x_i}^\mu$. Thus, the time evolution gives rise to a loop in position space:

$$Z = \int\limits_{\substack{{x_f}^\mu = {x_i}^\mu \\ {x_f}^\mu, {x_i}^\mu}} \mathcal{D}x^\mu \exp\left\{ \frac{\mathrm{i}}{\hbar} \int \mathrm{d}^d x^\mu \, \mathcal{L} \right\} \tag{9.18}$$

We can characterise the loop ${x_f}^\mu = {x_i}^\mu$ in terms of their homotopy class [50]. If two loops (or, in general in any topology, two paths) can be continuously deformed into each other, they're considered homotopic (or, equivalently, homotopically equivalent or in the same homotopy class); the set of all homotopy classes forms the fundamental group π_1. Since π_1 forms a transformation group on the measure x^μ, we can factorise the path integral by a one-dimensional complex-valued irreducible representation of π_1 (if such a representation exists, which it does):

$$Z = \sum_{\text{અ} \in \pi_1} \chi(\text{અ}) \int\limits_{\substack{{x_{\text{અ},f}}^\mu = {x_{\text{અ},i}}^\mu \\ {x_{\text{અ},f}}^\mu, {x_{\text{અ},i}}^\mu}} \mathcal{D}{x_{\text{અ}}}^\mu \exp\left\{ \frac{\mathrm{i}}{\hbar} \int \mathrm{d}^d {x_{\text{અ}}}^\mu \, \mathcal{L} \right\} \tag{9.19}$$

We can define the particles as a collection of hard / impenetrable particles by defining their positions $\vec{\mathbf{x}}_i \in \mathbb{R}^{d-1}$ as points excluded from the configuration space that Z is defined over:

$$\Delta := \left\{ (\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_n) \in \mathbb{R}^{d-1} : \vec{\mathbf{x}}_i = \vec{\mathbf{x}}_j \text{ for } i \neq j \right\} \tag{9.20}$$

In terms of Δ, then, the configuration space is defined as $(\mathbb{R}^{d-1} - \Delta)/S_n$, where S_n is the permutation group, and the corresponding fundamental group $\pi_1(S_n)$ is given by [58–60]:

$$M_n := \frac{\mathbb{R}^{d-1} - \Delta}{S_n}, \qquad \pi_1(M_n) = \begin{cases} S_n \text{ if } d \geq 3 \\ B_n \text{ if } d = 2 \end{cases} \tag{9.21}$$

Here, B_n is the braid group. In $d \geq 3$, we have $\chi(\text{અ}) = \pm 1$; i.e., the system supports either bosons or fermions. However, for $d = 2$, we have $\chi(\text{અ}) = \mathrm{e}^{\mathrm{i}\theta}$ for $\theta \in \mathbb{R}$: the particle statistics interpolates continuously between bosons and fermions. (We note that for $d = 1$, we can map bosons and fermions to each other [61, 62]. Additionally, if we didn't impose "hardness" on the particles (i.e., if we didn't remove Δ), then the particles would simply be bosons.) We can specifically derive $\chi(\text{અ})$ to be in terms of the angle that one particle loops around another:

$$\chi(\text{અ}) = \exp\left\{ \mathrm{i}\nu \sum_{i<j} \int_{t_i}^{t_f} \mathrm{d}\text{ટ} \, \frac{\mathrm{d}\phi_{ij}}{\mathrm{d}\text{ટ}} \right\} \tag{9.22}$$

Here, ϕ_{ij} is the phase that each particle picks up while encircling the other; this is, in fact, the Berry phase. Meanwhile, ν is related to the genus of the manifold that our 2D surface is defined over. (As always, we leave the derivation of (9.22), and the demonstration that soft particles reduce to bosons, for

the unabridged version.)

A Extra Derivations

A.1 Cauchy-Riemann Equations and the Scale Factor

For a conformally invariant system with $\varphi^* g = \Lambda g$, We examine the infinitesimal coordinate transformations $x^\mu \mapsto y^\mu$ up to first order in $\epsilon(x) \ll 1$. To this order, we have y given by $y^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)$. Applying this to the conformality requirement (1.1) for $g_{\mu\nu} \mapsto g_{\rho\sigma}$, we have:

$$\begin{aligned} g_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} &= g_{\rho\sigma}\frac{\partial\big[x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)\big]}{\partial x^\mu}\frac{\partial\big[x^\sigma + \epsilon^\sigma(x) + \mathcal{O}(\epsilon^2)\big]}{\partial x^\nu} \\ &= g_{\rho\sigma}\left(\frac{\partial x^\rho}{\partial x^\mu} + \frac{\partial \epsilon^\rho}{\partial x^\mu} + \mathcal{O}(\epsilon^2)\right)\left(\frac{\partial x^\sigma}{\partial x^\nu} + \frac{\partial \epsilon^\sigma}{\partial x^\nu} + \mathcal{O}(\epsilon^2)\right) = g_{\rho\sigma}\left(\delta^\rho_\mu + \frac{\partial \epsilon^\rho}{\partial x^\mu}\right)\left(\delta^\sigma_\nu + \frac{\partial \epsilon^\sigma}{\partial x^\nu}\right) + \mathcal{O}(\epsilon^2) \\ &= g_{\rho\sigma}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\mu\frac{\partial \epsilon^\sigma}{\partial x^\nu} + \delta^\sigma_\nu\frac{\partial \epsilon^\rho}{\partial x^\mu} + \underbrace{\frac{\partial \epsilon^\rho}{\partial x^\mu}\frac{\partial \epsilon^\sigma}{\partial x^\nu}}_{\mathcal{O}(\epsilon^2)}\right) + \mathcal{O}(\epsilon^2) \\ &= g_{\rho\sigma}\,\delta^\rho_\mu\,\delta^\sigma_\nu + g_{\rho\sigma}\,\delta^\rho_\mu\frac{\partial \epsilon^\sigma}{\partial x^\nu} + g_{\rho\sigma}\,\delta^\sigma_\nu\frac{\partial \epsilon^\rho}{\partial x^\mu} + \mathcal{O}(\epsilon^2) = g_{\mu\nu} + g_{\mu\sigma}\frac{\partial \epsilon^\sigma}{\partial x^\nu} + g_{\rho\nu}\frac{\partial \epsilon^\rho}{\partial x^\mu} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{A.1}$$

Now, using the fact that $g_{\mu\nu}$ is constant with respect to ∂_μ, we can bring $g_{\mu\nu}$ inside the derivatives:

$$g_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = g_{\mu\nu} + \frac{\partial\big[g_{\mu\sigma}\epsilon^\sigma\big]}{\partial x^\nu} + \frac{\partial\big[g_{\rho\nu}\epsilon^\rho\big]}{\partial x^\mu} + \mathcal{O}(\epsilon^2) \tag{A.2}$$

Finally, applying $\epsilon_\mu = g_{\mu\nu}\,\epsilon^\nu$, this gives:

$$g_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = g_{\mu\nu} + \frac{\partial\big[g_{\mu\sigma}\,\epsilon^\sigma\big]}{\partial x^\nu} + \frac{\partial\big[g_{\rho\nu}\,\epsilon^\rho\big]}{\partial x^\mu} + \mathcal{O}(\epsilon^2) = g_{\mu\nu} + \frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu} + \mathcal{O}(\epsilon^2) \tag{A.3}$$

We want to find the conditions under which this is equivalent to the conformality requirement given in (1.1); *i.e.*, we want to find the conditions that up to $\mathcal{O}(\epsilon^2)$ satisfy:

$$g_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = g_{\mu\nu} + \left(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}\right) = \Lambda(x)\,g_{\mu\nu} \tag{A.4}$$

If we write $\partial_\nu\,\epsilon_\mu + \partial_\mu\,\epsilon_\nu = K(x)\,g_{\mu\nu}$ for some scalar function $K(x)$, we can write the above expression as:

$$g_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = g_{\mu\nu} + \left(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}\right) + \mathcal{O}(\epsilon^2) = g_{\mu\nu} + K(x)\,g_{\mu\nu} = (1 + K(x))\,g_{\mu\nu} = \Lambda(x)\,g_{\mu\nu} \tag{A.5}$$

This gives the constraint:

$$\boxed{1 + K(x) = \Lambda(x)} \tag{A.6}$$

To actually *determine* $K(x)$, we trace out $\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = K(x)\, g_{\mu\nu}$ by contracting it with $g^{\mu\nu}$:

$$g^{\mu\nu}\big(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\big) = K(x)\, g^{\mu\nu}\, g_{\mu\nu} \tag{A.7}$$

Evaluating each side, and using $\delta^\mu_\mu = d$ where d is the dimension of the theory, we have:

$$\begin{gathered} g^{\mu\nu}\big(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\big) = \partial^\mu \epsilon_\mu + \partial^\nu \epsilon_\nu = \partial^\mu \epsilon_\mu + \partial^\mu \epsilon_\mu = 2\,\partial^\mu \epsilon_\mu \\ K(x)\, g^{\mu\nu}\, g_{\mu\nu} = K(x)(\delta^\mu_\mu) = K(x) \cdot d \end{gathered} \tag{A.8}$$

This gives $\eta^{\mu\nu}\big(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\big) = K(x)\, g^{\mu\nu}\, g_{\mu\nu}$ as $2\,\partial^\mu \epsilon_\mu = K(x) \cdot d$, so we have $K(x)$ as:

$$\boxed{K(x)\, K(x) = \frac{2\,\partial^\mu \epsilon_\mu}{d}} \tag{A.9}$$

From this, we have that the transformation $y^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)$ is conformal when we have:

$$\boxed{\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = \frac{2\,\partial^\mu \epsilon_\mu}{d} = \frac{2(\partial \cdot \epsilon)}{d} g_{\mu\nu}} \tag{A.10}$$

(This is entirely unsurprising: these are just the Cauchy-Riemann equations.) This then gives the scale factor to order $\mathcal{O}(\epsilon)$ as:

$$\boxed{\Lambda(x) = 1 + K(x) = 1 + \frac{2\,\partial^\mu \epsilon_\mu}{d}} \tag{A.11}$$

These directly lead us to two famous identities. For the first, we start by taking the contravariant derivative ∂^ν of the $\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu$ relation and contracting over ν. This gives:

$$\begin{gathered} \partial^\nu\big[\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\big] = \partial^\nu\left[\frac{2(\partial \cdot \epsilon)}{d}\, g_{\mu\nu}\right] \\ \partial^\nu\, \partial_\nu \epsilon_\mu + \partial^\nu\, \partial_\mu \epsilon_\nu = \frac{2}{d}\, \underbrace{\left(\partial^\nu\, g_{\mu\nu}\right)}_{\partial_\mu} (\partial \cdot \epsilon) = \frac{2}{d}\, \partial_\mu (\partial \cdot \epsilon) \end{gathered} \tag{A.12}$$

(Here, we used the fact that $(\partial \cdot \epsilon)$ is a scalar with respect to ∂^ν.) Since we're in Minkowski space, the Riemann tensor $R^\lambda_{\sigma\mu\nu}$ given by $R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma}$ vanishes everywhere, and thus the covariant derivatives ∂_μ equal partial derivatives $\vec{\nabla}$. Thus, $\big[\partial_\mu, \partial_\nu\big] = 0$, and so we have:

$$[\partial^\mu, \partial_\nu] = [g^{\mu\alpha}\, \partial_\alpha, \partial_\nu] = [g^{\mu\alpha}\, \partial_\alpha, \partial_\nu] = g^{\mu\alpha}\, [\partial_\alpha, \partial_\nu] = 0 \tag{A.13}$$

This gives $\partial^\nu\, \partial_\nu \epsilon_\mu + \partial^\nu\, \partial_\mu \epsilon_\nu$ as:

$$\partial^\nu\, \partial_\nu \epsilon_\mu + \partial^\nu\, \partial_\mu \epsilon_\nu = \Box\, \epsilon_\mu + \partial_\mu\, \partial^\nu\, \epsilon_\nu = \Box\, \epsilon_\mu + \partial_\mu\, \partial^\nu\, \epsilon_\nu = \Box\, \epsilon_\mu + \partial_\mu (\partial \cdot \epsilon) \tag{A.14}$$

Plugging this into (A.12), we have:

$$\partial^\nu \partial_\nu \epsilon_\mu + \partial^\nu \partial_\mu \epsilon_\nu = \frac{2}{d}\partial_\mu(\partial\cdot\epsilon)\,\Box\,\epsilon_\mu + \partial_\mu(\partial\cdot\epsilon) = \frac{2}{d}\partial_\mu(\partial\cdot\epsilon) \tag{A.15}$$

Taking the derivative of this with respect to ∂_ν now gives:

$$\begin{aligned}\partial_\nu\big[\Box\,\epsilon_\mu + \partial_\mu(\partial\cdot\epsilon)\big] &= \partial_\nu\Big[\frac{2}{d}\partial_\mu(\partial\cdot\epsilon)\Big]\\ \partial_\nu\,\Box\,\epsilon_\mu + \partial_\nu\partial_\mu(\partial\cdot\epsilon) &= \frac{2}{d}\partial_\nu\partial_\mu(\partial\cdot\epsilon)\end{aligned} \tag{A.16}$$

Interchanging $\nu \leftrightarrow \mu$ in the 2nd term on the LHS and on the RHS, and using $\Box = \partial^\alpha\partial_\alpha$ to commute ∂_ν with $\Box$, we get:

$$\Box\,\partial_\nu\epsilon_\mu + \partial_\mu\partial_\nu(\partial\cdot\epsilon) = \frac{2}{d}\partial_\mu\partial_\nu(\partial\cdot\epsilon) \tag{A.17}$$

Then, exchanging $\mu \leftrightarrow \nu$ in (A.17), we get:

$$\Box\,\partial_\mu\epsilon_\nu + \partial_\nu\partial_\mu(\partial\cdot\epsilon) = \frac{2}{d}\partial_\nu\partial_\mu(\partial\cdot\epsilon) \tag{A.18}$$

Then, adding (A.18) and (A.17) together, we get:

$$\begin{aligned}\Box\,\partial_\nu\epsilon_\mu + \partial_\mu\partial_\nu(\partial\cdot\epsilon) + \Box\,\partial_\mu\epsilon_\nu + \partial_\nu\partial_\mu(\partial\cdot\epsilon) &= \frac{2}{d}\partial_\mu\partial_\nu(\partial\cdot\epsilon) + \frac{2}{d}\partial_\nu\partial_\mu(\partial\cdot\epsilon)\\ (\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)(\partial\cdot\epsilon) + \Box(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) &= \frac{2}{d}(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)(\partial\cdot\epsilon)\end{aligned} \tag{A.19}$$

Using $[\partial_\mu, \partial_\nu] = 0$, we have $\partial_\mu\partial_\nu + \partial_\nu\partial_\mu = \partial_\mu\partial_\nu + \partial_\mu\partial_\nu = 2\,\partial_\mu\partial_\nu$. Applying this and (A.10), we have:

$$\begin{aligned}(\cancel{2}\,\partial_\mu\partial_\nu)(\partial\cdot\epsilon) + \Box\Big(\frac{\cancel{2}}{d}(\partial\cdot\epsilon)\,g_{\mu\nu}\Big) &= \frac{\cancel{2}}{d}(2\,\partial_\mu\partial_\nu)(\partial\cdot\epsilon)\\ d\,\partial_\mu\partial_\nu(\partial\cdot\epsilon) + g_{\mu\nu}\,\Box(\partial\cdot\epsilon) - 2\,\partial_\mu\partial_\nu(\partial\cdot\epsilon) &= 0\\ (d-2)\partial_\mu\partial_\nu(\partial\cdot\epsilon) + g_{\mu\nu}\,\Box(\partial\cdot\epsilon) &= 0\\ \big[(d-2)\partial_\mu\partial_\nu + g_{\mu\nu}\,\Box\big](\partial\cdot\epsilon) &= 0\end{aligned} \tag{A.20}$$

Before continuing, we note something quite important about this equation, following [2]. The $d-2$ term highlights that the identity we get from further manipulations will hold specifically for $d \geq 3$. 2D CFTs, on the other hand, are incredibly special; this expression is just one example. ([2] also discusses the $d = 1$ case; i.e., conformal quantum mechanics.) Contracting this with $g^{\mu\nu}$, we get:

$$\begin{aligned}g_{\mu\nu}\big[(d-2)\partial_\mu\partial_\nu + g_{\mu\nu}\,\Box\big](\partial\cdot\epsilon) &= 0\\ \big((d-2)g^{\mu\nu}\,\partial_\mu\partial_\nu + g^{\mu\nu}\,g_{\mu\nu}\,\Box\big)(\partial\cdot\epsilon) &= 0\end{aligned} \tag{A.21}$$

Applying $g^{\mu\nu}\partial_\mu = \partial^\nu$ and $g^{\mu\nu}\,g_{\mu\nu} = \delta^\mu_\mu = d$, this expression is simply:

$$\begin{gathered}((d-2)\partial^\nu\partial_\nu + d\,\Box)(\partial\cdot\epsilon) = ((d-2)\Box + d\,\Box)(\partial\cdot\epsilon) = (2d-2)\Box(\partial\cdot\epsilon) = 0 \\ \boxed{(d-1)\Box(\partial\cdot\epsilon) = 0}\end{gathered} \tag{A.22}$$

(Again, this is specifically for $d \geq 3$.) For the second identity, we take the derivative of (A.10) with respect to ∂_ρ:

$$\begin{aligned}\partial_\rho\left[\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu\right] &= \partial_\rho\left[\frac{2\,(\partial\cdot\epsilon)}{d}\,g_{\mu\nu}\right] \\ \partial_\rho\partial_\nu\epsilon_\mu + \partial_\rho\partial_\mu\epsilon_\nu &= \frac{2}{d}\,g_{\mu\nu}\,\partial_\rho[\partial\cdot\epsilon] + \frac{2}{d}(\partial\cdot\epsilon)\,\partial_\rho\left[g_{\mu\nu}\right]\end{aligned} \tag{A.23}$$

Since we're in flat space, we have $\partial_\rho\left[g_{\mu\nu}\right] = 0$, so this gives:

$$\partial_\rho\partial_\nu\epsilon_\mu + \partial_\rho\partial_\mu\epsilon_\nu = \frac{2}{d}\,g_{\mu\nu}\,\partial_\rho[\partial\cdot\epsilon] \tag{A.24}$$

We can take cyclic rotations of the indices:

$$\partial_\mu\partial_\rho\epsilon_\nu + \partial_\nu\partial_\rho\epsilon_\mu = \frac{2}{d}\,g_{\rho\mu}\,\partial_\nu[\partial\cdot\epsilon] \tag{A.25}$$

$$\partial_\nu\partial_\mu\epsilon_\rho + \partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}\,g_{\nu\rho}\,\partial_\mu[\partial\cdot\epsilon] \tag{A.26}$$

Then, (A.25) plus (A.26) minus (A.24) gives:

$$\begin{aligned}&\partial_\mu\partial_\rho\epsilon_\nu + \partial_\nu\partial_\rho\epsilon_\mu + \partial_\nu\partial_\mu\epsilon_\rho + \partial_\mu\partial_\nu\epsilon_\rho - \partial_\rho\partial_\nu\epsilon_\mu - \partial_\rho\partial_\mu\epsilon_\nu \\ &\qquad = \frac{2}{d}\,g_{\rho\mu}\,\partial_\nu[\partial\cdot\epsilon] + \frac{2}{d}\,\eta_{\nu\rho}\,\partial_\mu[\partial\cdot\epsilon] - \frac{2}{d}\,\eta_{\mu\nu}\,\partial_\rho[\partial\cdot\epsilon]\end{aligned} \tag{A.27}$$

Applying $\left[\partial_\alpha, \partial_\beta\right] = 0$ and regrouping to collect the stuff that cancels, we can rewrite the LHS as:

$$\partial_\rho\partial_\mu\epsilon_\nu - \partial_\rho\partial_\mu\epsilon_\nu + \partial_\rho\partial_\nu\epsilon_\mu - \partial_\rho\partial_\nu\epsilon_\mu + \partial_\mu\partial_\nu\epsilon_\rho + \partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}(g_{\rho\mu}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu - g_{\mu\nu}\,\partial_\rho)[\partial\cdot\epsilon] \tag{A.28}$$

The first four terms on the LHS cancel, giving our other identity:

$$\begin{gathered}2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}(g_{\rho\mu}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu - g_{\mu\nu}\,\partial_\rho)[\partial\cdot\epsilon] \\ \boxed{\partial_\mu\partial_\nu\epsilon_\rho = \frac{1}{d}(g_{\rho\mu}\,\partial_\nu + g_{\nu\rho}\,\partial_\mu - g_{\mu\nu}\,\partial_\rho)[\partial\cdot\epsilon]}\end{gathered} \tag{A.29}$$

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