Corregidum to “Novel whitening approaches in functional settings”

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1. Due to production errors, Equation 3 in pp. 3 is written as

$$\langle f, g\rangle_{\mathbb{M}} = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle f, \gamma_j \rangle \langle g, \gamma_j \rangle = \left\langle \Gamma^{1/2\dagger} f, \Gamma^{1/2\dagger} g \right\rangle f, \quad g \in \mathbb{M},$$

   while it was originally written as

$$\langle f, g\rangle_{\mathbb{M}} = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle f, \gamma_j \rangle \langle g, \gamma_j \rangle = \left\langle \Gamma^{1/2\dagger} f, \Gamma^{1/2\dagger} g \right\rangle \quad f, g \in \mathbb{M}.$$

2. In §3, the statement reads, “Then, we can use the inner product (3) to construct a space of isotropic functions (i.e., their covariance operator satisfies the identity), so that the space ends up having a certain Gaussian appearance...” The term *space of isotropic functions* is unclear in the current context. Since our whitening operators are mappings defined through elements over T, this does not necessarily imply that the realizations of $\mathbb{X}$ are on the unit sphere $S = \{f \in \mathbb{M} \mid \|f\|^2 = 1\}$. The isotropy property would be satisfied when whitening the basis expansion coefficients in the direction of its transpose, assuming dependencies in a secondary domain exist. Therefore, one could use the following instead: “....to construct *a space of whitened functions* (i.e., their covariance operator satisfies the identity), so that the space ends up having a certain Gaussian appearance...” .

3. In §4, the sentence “As 2tr $(\Gamma_{X\mathbb{X}})$ is the only dependence between the original and the whitened variable, the minimization problem can be reduced to the maximization of tr $(\Gamma_{X\mathbb{X}})$.” reads also as “.. is the only *dependent term*...”.

4. Note that, in §4 the term $\mathrm{tr}(\Gamma_{\mathbb{X}})$ in the quadratic distances diverges (the trace of $\Gamma_{\mathbb{X}}$ is an infinite sum of ones). However, $\mathrm{tr}(\Gamma_{\mathbb{X}})$ is not accounted for in the proof. In order for these distances to converge, one has to consider regularization or finite space dependency. Furthermore, the operator $\Gamma_{X\mathbb{X}}$ coincides with $\Gamma^{1/2}$ if $\Psi \equiv \Gamma^{1/2\dagger}$. Note we only know that this operator belongs to the class of Hilbert Schmidt operators (from the trace property of the autocovariance operator), but this fact does not necessarily imply $\Gamma^{1/2}$ has finite trace. Hence, we further assume that under mild conditions, $\mathrm{tr}(\Gamma^{1/2}) < \infty$ is satisfied.

5. In the Technical proofs (first paragraph), due to abuse of notation, in the sentence “Note that Condition 1 cannot be reached when $\langle X, \gamma_j\rangle^2 = \lambda_j$, or for $c_j \to c > 0$, $\langle X, \gamma_j\rangle^2 = \lambda_j c_j$...”, X stands for a deterministic function.

6. In the technical proof of Proposition 1, it is stated that “The operator $P_{\overline{\mathrm{ran}}(\Gamma^{1/2})}$ is compact...”. However, this characterization of the projection operator as compact appears to be mistaken since projection operators are typically not compact in infinite-dimensional spaces.

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CHARACTERIZATION OF $\mathcal{R} = \mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger}$

Suppose X is expanded as $X = \sum_{k=1}^{\infty}\langle X, e_k\rangle e_k$ and note the following:

$$\mathcal{V} = \sum_{k=1}^{\infty}\mathcal{P}_{e_k}\Gamma\mathcal{P}_{e_k} = \sum_{k=1}^{\infty}\langle\Gamma e_k, e_k\rangle\mathcal{P}_{e_k} = \sum_{k=1}^{\infty}E(\langle X, e_k\rangle^2)(e_k\otimes e_k) = \sum_{k=1}^{\infty}\eta_k(e_k\otimes e_k). \tag{1}$$

Now, consider the operator

$$\mathcal{R} \equiv E\left(\sum_{j=1}^{\infty}\frac{\langle X, e_j\rangle}{\eta_j^{1/2}}e_j\otimes\frac{\langle X, e_j\rangle}{\eta_j^{1/2}}e_j\right). \tag{2}$$

Observe that $\mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger} = \sum_{j=1}^{\infty}\eta_j^{-1/2}E(\mathcal{P}_{e_j}X\otimes\mathcal{P}_{e_j}X)\eta_j^{-1/2}$, where $\mathcal{V}^{\dagger}$ is the Moore-Penrose inverse of $\mathcal{V} = \sum_{j=1}^{\infty}\eta_j\mathcal{P}_{e_j}$. This shows that $\mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger}$ is equivalent to the operator $\mathcal{R}$, as defined in 2. Note this operator bears resemblance to the classical correlation matrix in the multivariate setting.

REFERENCES

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