HOMEWORK ASSIGNMENT 10 — VIVIANA MÁRQUEZ

Problem 10.1. Find an exampe of $A \in \mathrm{End}(\mathbb{C}^6)$ such that its characteristic polynomial is $(x-1)^2x^4$ and its minimal polynomial is $(x-1)^2x^2$.

Solution:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 10.2. Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Compute A^{20}.

Solution:

$$\begin{aligned} \chi(\lambda) = det(A - \lambda I) &= (2-\lambda)(-1-\lambda)(\lambda) - (1)(-1-\lambda)(-1) \\ &= -(\lambda-1)(\lambda-1)(\lambda+1) \end{aligned}$$

$\lambda_1 = -1$
$\lambda_2 = 1$ (multiplicity 2)

Let us now find the eigenspaces corresponding to the eigenvalues:

$\lambda_1 = -1$

$$Null(A-(-1)I) \Rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\lambda_2 = 1$

$$Null(A-I) \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Since we only obtained two linearly independent eigenvectors, then we find the following Jordan Canonical Form with two Jordan Blocks:

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} Av_1 = v_1 &\Rightarrow (A+I)v_1 = 0 \\ Av_2 = v_2 &\Rightarrow (A-I)v_2 = 0 \\ Av_3 = v_2 + v_3 &\Rightarrow (A-I)v_3 = v_2 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

So, the transition matrix C such that $A = CJC^{-1}$ is obtained:

$$C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad C^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vdots$$

$$J^{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 20 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $A^{20} = CJ^{20}C^{-1}$, then:

$$A^{20} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 20 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 20 \\ 0 & 1 & 0 \\ -20 & 0 & -19 \end{pmatrix}$$

Problem 10.3. Consider the vector space of complex polynomials of degree not greater than n

$$\mathbb{C}_n[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{C}, \forall i\}$$

and define $A \in \text{End}(\mathbb{C}_n[x])$ by the formula $(Ap)(x) = (\frac{d}{dx}p)(x)$, $p \in \mathbb{C}_n[x]$. Find the Jordan canonical form of the operator A.

Solution:

$$\mathbb{C}_n[x] \text{ is a vector space.}$$
$$\{1, x, x^2, x^3, \dots, x^n\} \text{ is a basis of } \mathbb{C}_n[x]$$

$(Ap)(x) = (\frac{d}{dx}p)(x)$

$A(1) = 0$
$A(x) = 1$
$A(x^2) = 2x$
$A(x^3) = 3x^2$
$\vdots$
$A(x^n) = nx^{n-1}$

We obtain the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & & \\ \vdots & 0 & 2 & & & \\ & & 0 & 3 & & \\ & & & 0 & \ddots & \\ \vdots & & & & \ddots & n \\ 0 & \dots & & & \dots & 0 \end{pmatrix}$$

Computing,

$$det \begin{pmatrix} -\lambda & 1 & 0 & \dots & & \\ \vdots & -\lambda & 2 & & & \\ & & -\lambda & 3 & & \\ & & & -\lambda & \ddots & \\ \vdots & & & & \ddots & n \\ 0 & \dots & & & \dots & -\lambda \end{pmatrix}$$

We find the eigenvalue $\lambda = 0$ that has multiplicity $n + 1$.

$$Null(A - 0I) \Rightarrow \begin{pmatrix} 0 & 1 & 0 & \dots & & \\ \vdots & 0 & 2 & & & \\ & & 0 & 3 & & \\ & & & 0 & \ddots & \\ \vdots & & & & \ddots & n \\ 0 & \dots & & & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Since we obtained only one independent eigenvector, then we have a single Jordan Block in the Jordan Canonical form.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & & \\ \vdots & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & \ddots & \\ \vdots & & & & \ddots & 1 \\ 0 & \dots & & & \dots & 0 \end{pmatrix}$$

Problem 10.4. Consider the vector space of real polynomials of degree not greater than 3

$$\mathbb{R}_3[x] = \{\sum_{i=0}^{3} a_i x^i \mid a_i \in \mathbb{R}, \forall i\}$$

and define $A \in \text{End}(\mathbb{R}_3[x])$ by the formula $(Ap)(x) = p(x+1)$, $p \in \mathbb{R}_3[x]$. Find all eigenvalues and eigenvectors of the operator A.

<u>*Solution:*</u>

$$\mathbb{R}_n[x] \text{ is a vector space.}$$
$$\{1, x, x^2, x^3\} \text{ is a basis of } \mathbb{C}_n[x]$$

$(Ap)(x) = p(x+1)$

$A(1) = 1 = 1$
$A(x) = (x+1) = x+1$
$A(x^2) = (x+1)^2 = x^2 + 2x + 1$
$A(x^3) = (x+1)^3 = x^3 + 3x^2 + 3x + 1$

We obtain the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using the characteristic polynomial, we find the eigenvalue $\lambda = 1$ with multiplicity 4.

Now, let us find its corresponding eigenvector:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Problem 10.5*. Consider the vector space of complex polynomials of degree not greater than n

$$\mathbb{C}_n[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{C}, \forall i\}$$

and define $A \in \mathrm{End}(\mathbb{C}_n[x])$ by the formula $(Ap)(x) = 5p(x+1) - 3p(x)$, $p \in \mathbb{C}_n[x]$. Find the Jordan canonical form of the operator A.

Solution:

$$\mathbb{C}_n[x] \text{ is a vector space.}$$
$$\{1, x, x^2, x^3, \ldots, x^n\} \text{ is a basis of } \mathbb{C}_n[x]$$

$(Ap)(x) = 5p(x+1) - 3p(x)$

$A(1) = 2$
$A(x) = 5(x+1) - 3(x) = 2x + 5$
$A(x^2) = 5(x+1)^2 - 3(x)^2 = 2x^2 + 10x + 5$
$A(x^3) = 5(x+1)^3 - 3(x)^3 = 2x + 15x^2 + 15x + 5$
$\vdots$
$A(x^n) = 5(x+1)^n - 3(x)^n$

We obtain the following matrix:

$$\begin{pmatrix} 2 & 5 & 5 & 5 & \ldots & \\ 0 & 2 & 10 & 15 & & \\ 0 & 0 & 2 & 15 & & \\ 0 & 0 & 0 & 2 & \ddots & \\ \vdots & & & & \ddots & \ddots \\ 0 & \ldots & & & \ldots & 2 \end{pmatrix}$$

Using the characteristic polynomial, we find the eigenvalue $\lambda = 2$ with multiplicity $n+1$.
Let us now find the eigenspace corresponding to the eigenvalue:

$$Null(A - 2I) \Rightarrow \begin{pmatrix} 0 & 5 & 5 & \ldots & & \\ \vdots & 0 & 10 & & & \\ & & 0 & 15 & & \\ & & & 0 & \ddots & \\ \vdots & & & & \ddots & n \\ 0 & \ldots & & & \ldots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Since we obtained only one independent eigenvector, then we have a single Jordan Block in the Jordan Canonical form.

$$\begin{pmatrix} 2 & 1 & 0 & \ldots & & \\ 0 & 2 & 1 & & & \\ \vdots & & 2 & 1 & & \\ & & & 2 & \ddots & \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & & & \ldots & 2 \end{pmatrix}$$